# Chapter 7
# Known Functions of Fixed Effects

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In previous chapters we have dealt with linear relationships among $\boldsymbol{\beta}$ of the following types.

**1.** $\mathbf{M}'\boldsymbol{\beta}$ is a set of p-r non-estimable functions of $\boldsymbol{\beta}$ , and a solution to GLS or mixed model equations is obtained such that $\mathbf{M}'\boldsymbol{\beta}^o = \mathbf{c}$.

**2.** $\mathbf{K}'\boldsymbol{\beta}$ is a set of r estimable functions. Then we write a set of equations, the solution to which yields directly BLUE of $\mathbf{K}'\boldsymbol{\beta}$.

**3.** $\mathbf{H}'\boldsymbol{\beta}$ is a set of estimable functions that are used in hypothesis testing.

In this chapter we shall be concerned with defined linear relationships of the form,

$$\mathbf{T}'\boldsymbol{\beta} = \mathbf{c}.$$

All of these are linearly independent. The consequence of these relationships is that functions of $\boldsymbol{\beta}$ may become estimable that are not estimable under a model with no such definitions concerning $\boldsymbol{\beta}$ . In fact, if $\mathbf{T}'\boldsymbol{\beta}$ represents $p - r$ linearly independent non-estimable functions, all linear functions of $\boldsymbol{\beta}$ become estimable.

## 1 Tests of Estimability

If $\mathbf{T}'\boldsymbol{\beta}$ represents $t < p - r$ non-estimable functions the following rule can be used to determine what functions are estimable.

$$\begin{pmatrix} \mathbf{X} \\ \mathbf{T}' \end{pmatrix} \mathbf{C} = \mathbf{0}, \tag{1}$$

where $\mathbf{C}$ is a $p \times (r - t)$ matrix with rank $r - t$. $\mathbf{C}$ always exists. Then $\mathbf{K}'\boldsymbol{\beta}$ is estimable if and only if

$$\mathbf{K}'\mathbf{C} = \mathbf{0}. \tag{2}$$

To illustrate, suppose that

$$\mathbf{X} = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 3 & 5 \\ 1 & 3 & 4 & 0 \\ 3 & 2 & 5 & 7 \\ 1 & 1 & 2 & 2 \\ 2 & 1 & 3 & 5 \end{pmatrix}.$$

with $p = 4$, and $r = 2$ becuase

$$\mathbf{X} \begin{pmatrix} 1 & 3 \\ 1 & -1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} = \mathbf{0}$$

Suppose we define $\mathbf{T}' = (1\ 2\ 2\ 1)$. This is a non-estimable function because

$$(1\ 2\ 2\ 1) \begin{pmatrix} 1 & 3 \\ 1 & -1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} = (1 \quad 0) \neq \mathbf{0}.$$

Now

$$\begin{pmatrix} \mathbf{X} \\ \mathbf{T}' \end{pmatrix} \mathbf{C} = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 3 & 5 \\ 1 & 3 & 4 & 0 \\ 3 & 2 & 5 & 7 \\ 1 & 1 & 2 & 2 \\ 2 & 1 & 3 & 5 \\ 1 & 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 0 \\ -1 \end{pmatrix} = \mathbf{0}.$$

Therefore $\mathbf{K}'\boldsymbol{\beta}$ is estimable if and only if $\mathbf{K}'(3 \ -1 \ \ 0 \ -1)' = \mathbf{0}$. If we had defined

$$\mathbf{T}' = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 3 \end{pmatrix},$$

any function of $\boldsymbol{\beta}$ would be estimable because rank $\begin{pmatrix} \mathbf{X} \\ \mathbf{T}' \end{pmatrix} = 4$. This is because $p - r = 4 - 2$ non-estimable functions are defined.

## 2 BLUE when $\beta$ Subject to $\mathbf{T}'\beta$

One method for computing BLUE of $\mathbf{K}'\boldsymbol{\beta}$, estimable given $\mathbf{T}'\boldsymbol{\beta} = \mathbf{c}$, is $\mathbf{K}'\boldsymbol{\beta}^o$, where $\boldsymbol{\beta}^o$ is a solution to either of the following.

$$\begin{pmatrix} \mathbf{X}'\mathbf{V}^{-1}\mathbf{X} & \mathbf{T} \\ \mathbf{T}' & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}^o \\ \boldsymbol{\theta} \end{pmatrix} = \begin{pmatrix} \mathbf{X}'\mathbf{V}^{-1}\mathbf{y} \\ \mathbf{c} \end{pmatrix}. \tag{3}$$

$$\begin{pmatrix} \mathbf{X}'\mathbf{R}^{-1}\mathbf{X} & \mathbf{X}'\mathbf{R}^{-1}\mathbf{Z} & \mathbf{T} \\ \mathbf{Z}'\mathbf{R}^{-1}\mathbf{X} & \mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z} + \mathbf{G}^{-1} & \mathbf{0} \\ \mathbf{T}' & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}^o \\ \hat{\mathbf{u}} \\ \boldsymbol{\theta} \end{pmatrix} = \begin{pmatrix} \mathbf{X}'\mathbf{R}^{-1}\mathbf{y} \\ \mathbf{Z}'\mathbf{R}^{-1}\mathbf{y} \\ \mathbf{c} \end{pmatrix}. \tag{4}$$

If $\mathbf{T}'\boldsymbol{\beta}$ represents $p - r$ linearly independent non-estimable functions, $\boldsymbol{\beta}^o$ has a unique solution. A second method where $\mathbf{c} = \mathbf{0}$ is the following. Partition $\mathbf{T}'$, with re-ordering of columns if necessary, as

$$\mathbf{T}' = [\mathbf{T}_1' \ \ \mathbf{T}_2'],$$

the re-ordering done, if necessary, so that $\mathbf{T}_2'$ is non-singular. This of course implies that $\mathbf{T}_2$ is square. Partition $\mathbf{X} = [\mathbf{X}_1 \ \ \mathbf{X}_2]$, where $\mathbf{X}_2$ has the same number of columns as $\mathbf{T}_2'$ and with the same re-ordering of columns as in $\mathbf{T}'$. Let

$$\mathbf{W} = \mathbf{X}_1 - \mathbf{X}_2(\mathbf{T}_2')^{-1}\mathbf{T}_1'.$$

Then solve for $\boldsymbol{\beta}^o$, in either of the following two forms.

$$\mathbf{W}'\mathbf{V}^{-1}\mathbf{W}\boldsymbol{\beta}_1^o = \mathbf{W}'\mathbf{V}^{-1}\mathbf{y}. \tag{5}$$

$$\begin{pmatrix} \mathbf{W}'\mathbf{R}^{-1}\mathbf{W} & \mathbf{W}'\mathbf{R}^{-1}\mathbf{Z} \\ \mathbf{Z}'\mathbf{R}^{-1}\mathbf{W} & \mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z} + \mathbf{G}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}_1^o \\ \hat{\mathbf{u}} \end{pmatrix} = \begin{pmatrix} \mathbf{W}'\mathbf{R}^{-1}\mathbf{y} \\ \mathbf{Z}'\mathbf{R}^{-1}\mathbf{y} \end{pmatrix} \tag{6}$$

In terms of the model with no definitions on the parameters,

$$E(\boldsymbol{\beta}_1^o) = (\mathbf{W}'\mathbf{V}^{-1}\mathbf{W})^{-}\mathbf{W}'\mathbf{V}^{-1}\mathbf{X}\boldsymbol{\beta}. \tag{7}$$

$$\boldsymbol{\beta}_2^o = -(\mathbf{T}_2')^{-1}\mathbf{T}_1'\boldsymbol{\beta}_1^o. \tag{8}$$

$$E(\boldsymbol{\beta}_2^o) = -(\mathbf{T}_2')^{-1}\mathbf{T}_1'\mathrm{E}(\boldsymbol{\beta}_1^o). \tag{9}$$

Let us illustrate with the same $\mathbf{X}$ used for illustrating estimability when $\mathbf{T}'\boldsymbol{\beta}$ is defined. Suppose we define

$$\mathbf{T}' = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 3 \end{pmatrix}, \ \mathbf{c} = \mathbf{0}.$$

then $\mathbf{T}'\boldsymbol{\beta}$ are non-estimable functions. Consequently the following GLS equations have a unique solution. It is assumed that $Var(\mathbf{y}) = \mathbf{I}\sigma_e^2$. The equations are

$$\sigma_e^{-2} \begin{pmatrix} 20 & 16 & 36 & 44 & 1 & 2 \\ 16 & 20 & 36 & 28 & 2 & 1 \\ 36 & 36 & 72 & 72 & 2 & 1 \\ 44 & 28 & 72 & 104 & 1 & 3 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}^o \\ \boldsymbol{\theta} \end{pmatrix} = \begin{pmatrix} 46 \\ 52 \\ 98 \\ 86 \\ 0 \\ 0 \end{pmatrix} \sigma_e^{-2}. \tag{10}$$

and $\mathbf{y}' = [5,\ 3,\ 7,\ 2,\ 6,\ 8]$.

The solution is $[380,\ -424,\ 348,\ -228,\ 0,\ 0]/72$. If $\mathbf{T}'\boldsymbol{\beta} = \mathbf{0}$ is really true, any linear function of $\boldsymbol{\beta}$ is estimable.

By the method of (5)

$$\mathbf{X}_1' = \begin{pmatrix} 1 & 2 & 1 & 3 & 1 & 2 \\ 2 & 1 & 3 & 2 & 1 & 1 \end{pmatrix},$$

$$\mathbf{X}_2' = \begin{pmatrix} 3 & 3 & 4 & 5 & 2 & 3 \\ 1 & 5 & 0 & 7 & 2 & 5 \end{pmatrix},$$

$$\mathbf{T}_1' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \mathbf{T}_2' = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix},$$

then

$$\mathbf{W}' = \begin{pmatrix} -.2 & -1.6 & .2 & -2.2 & -.6 & -1.6 \\ -1.0 & -2.0 & -1.0 & -3.0 & -1.0 & -2.0 \end{pmatrix}.$$

Equations like (5) are

$$\sigma_e^{-2} \begin{pmatrix} 10.4 & 13.6 \\ 13.6 & 20.0 \end{pmatrix} \boldsymbol{\beta}_1^o = \begin{pmatrix} -25.2 \\ -46.0 \end{pmatrix} \sigma_e^{-2}.$$

The solution is $\boldsymbol{\beta}_1^o = (380, \text{-}424)/72$. By (8)

$$\boldsymbol{\beta}_2^o = -\begin{pmatrix} .2 & 1.0 \\ .6 & 0 \end{pmatrix} \begin{pmatrix} 380 \\ -424 \end{pmatrix} /72 = \begin{pmatrix} 348 \\ -228 \end{pmatrix} /72.$$

These are identical to the result by method (3). $E(\boldsymbol{\beta}_1^o)$ by (7) is

$$\begin{pmatrix} 0 & 2.5 & 2.5 & -2.5 \\ -1.0 & -2.5 & -3.5 & -.5 \end{pmatrix} \boldsymbol{\beta}.$$

It is easy to verify that these are estimable under the restricted model.

At this point it should be noted that the computations under $\mathbf{T}'\boldsymbol{\beta} = \mathbf{c}$, where these represent $p-r$ non-estimable functions are identical with those previously described where the GLS or mixed model equation solution is restricted to $\mathbf{M}'\boldsymbol{\beta}^o = \mathbf{c}$. However, all linear functions of $\boldsymbol{\beta}$ are estimable under the restriction regarding parameters whereas they are not when these restrictions are on the solution, $\boldsymbol{\beta}^o$. Restrictions $\mathbf{M}'\boldsymbol{\beta}^o = \mathbf{c}$ are used only for convenience whereas $\mathbf{T}'\boldsymbol{\beta} = \mathbf{c}$ are used because that is part of the model.

Now let us illustrate with our same example, but with only one restriction, that being

$$(2 \ \ 1 \ \ 1 \ \ 3)\ \boldsymbol{\beta} = 0.$$

Then equations like (3) are

$$\sigma_e^{-2}\begin{pmatrix} 20 & 16 & 36 & 44 & 2 \\ 16 & 20 & 36 & 28 & 1 \\ 36 & 36 & 72 & 72 & 1 \\ 44 & 28 & 72 & 104 & 3 \\ 2 & 1 & 1 & 3 & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{\beta}^o \\ \boldsymbol{\theta} \end{pmatrix} = \begin{pmatrix} 46 \\ 52 \\ 98 \\ 86 \\ 0 \end{pmatrix}\sigma_e^{-2}.$$

These do not have a unique solution, but one solution is (-88, 0, 272, -32, 0)/144. By the method of (5)

$$\begin{aligned} \mathbf{T}_1' &= (2 \;\; 1 \;\; 1), \\ \mathbf{T}_2' &= 3. \end{aligned}$$

This leads to

$$9^{-1}\sigma_e^{-2}\begin{pmatrix} 68 & 52 & -32 \\ 52 & 116 & 128 \\ -32 & 128 & 320 \end{pmatrix}\boldsymbol{\beta}_1^o = \begin{pmatrix} -102 \\ 210 \\ 624 \end{pmatrix} 9^{-1}\sigma_e^{-2}.$$

These do not have a unique solution but one solution is (-88 0 272)/144 as in the other method for $\boldsymbol{\beta}_1^o$.

## 3 Sampling Variances

If the method of (3) is used,

$$Var(\mathbf{K}'\boldsymbol{\beta}^o) = \mathbf{K}'\mathbf{C}_{11}\mathbf{K}, \tag{11}$$

where $\mathbf{C}_{11}$ is the upper $p^2$ submatrix of a g-inverse of the coefficient matrix. The same is true for (4).

If the method of (5) is used

$$\begin{aligned} Var(\mathbf{K}_1'\boldsymbol{\beta}_1^o) &= \mathbf{K}_1'(\mathbf{W}'\mathbf{V}^{-1}\mathbf{W})^-\mathbf{K}_1. & (12) \\ Cov(\mathbf{K}_1'\boldsymbol{\beta}_1^o, \boldsymbol{\beta}_2^{o'}\mathbf{K}_2) &= -\mathbf{K}_1'(\mathbf{W}'\mathbf{V}^{-1}\mathbf{W})^-\mathbf{T}_1\mathbf{T}_2^{-1}\mathbf{K}_2. & (13) \\ Var(\mathbf{K}_2'\boldsymbol{\beta}_2^o) &= \mathbf{K}_2'(\mathbf{T}_2')^{-1}\mathbf{T}_1'(\mathbf{W}'\mathbf{V}^{-1}\mathbf{W})^-\mathbf{T}_1\mathbf{T}_2^{-1}\mathbf{K}_2. & (14) \end{aligned}$$

If the method of (6) is used, the upper part of a g-inverse of the coefficient matrix is used in place of (11).

Let us illustrate with the same example and with one restriction. A g-inverse of the coefficient matrix is

$$\frac{\sigma_e^2}{576}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 80 & -32 & -16 & 576 \\ 0 & -32 & 29 & 1 & -432 \\ 0 & -16 & 1 & 5 & 144 \\ 0 & 576 & -432 & 144 & 0 \end{pmatrix}.$$

Then

$$Var(\mathbf{K}'\boldsymbol{\beta}^o) = \frac{\sigma_e^2}{576} \ \mathbf{K}' \begin{pmatrix} 0 & 80 & -32 & -16 \\ 0 & -32 & 29 & 1 \\ 0 & -16 & 1 & 5 \end{pmatrix} \mathbf{K}.$$

Using the method of (5) a g-inverse of $\mathbf{W}'\mathbf{V}^{-1}\mathbf{W}$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 80 & -32 \\ 0 & -32 & 29 \end{pmatrix} \frac{\sigma_e^2}{576},$$

which is the same as the upper $3 \times 3$ of the matrix above. From (13)

$$-(\mathbf{W}'\mathbf{V}^{-1}\mathbf{W})^{-}\mathbf{T}_1\mathbf{T}_2^{-1} = -\frac{1}{576} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 80 & -32 \\ 0 & -32 & 29 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \frac{1}{3} = \begin{pmatrix} 0 \\ 16 \\ 1 \end{pmatrix} \frac{1}{576}$$

and for (14) $(\mathbf{T}_2')^{-1}\mathbf{T}_1'(\mathbf{W}'\mathbf{V}^{-1}\mathbf{W})^{-}\mathbf{T}_1\mathbf{T}_2^{-1} = 5/576$, thus verifying that the sampling variances are the same by the two methods.

## 4 Hypothesis Testing

As before let $\mathbf{H}_0'\boldsymbol{\beta} = \mathbf{c}_0$ be the null hypothesis and $\mathbf{H}_a'\boldsymbol{\beta} = \mathbf{c}_a$ be the alternative hypothesis, but now we have defined $\mathbf{T}'\boldsymbol{\beta} = \mathbf{c}$. Consequently $\mathbf{H}_0'\boldsymbol{\beta}$ and $\mathbf{H}_a'\boldsymbol{\beta}$ need be estimable only when $\mathbf{T}'\boldsymbol{\beta} = \mathbf{c}$ is assumed.

Then the tests proceed as in the unrestricted model except that for the null hypothesis computations we substitute

$$\begin{pmatrix} \mathbf{H}_0' \\ \mathbf{T}' \end{pmatrix} \boldsymbol{\beta} - \begin{pmatrix} \mathbf{c}_0 \\ \mathbf{c} \end{pmatrix} \text{ for } \mathbf{H}_0\boldsymbol{\beta} - \mathbf{c}_0. \tag{15}$$

and for the alternative hypothesis we substitute

$$\begin{pmatrix} \mathbf{H}_a' \\ \mathbf{T}' \end{pmatrix} \boldsymbol{\beta} - \begin{pmatrix} \mathbf{c}_a \\ \mathbf{c} \end{pmatrix} \text{ for } \mathbf{H}_a'\boldsymbol{\beta} - \mathbf{c}_a. \tag{16}$$

To illustrate suppose the unrestrained GLS equations are

$$\begin{pmatrix} 6 & 3 & 2 & 1 \\ 3 & 7 & 1 & 2 \\ 2 & 1 & 8 & 1 \\ 1 & 2 & 1 & 9 \end{pmatrix} \boldsymbol{\beta}^o = \begin{pmatrix} 9 \\ 12 \\ 15 \\ 16 \end{pmatrix}.$$

Suppose that we define $\mathbf{T}'\boldsymbol{\beta} = 0$ where $\mathbf{T}' = (3\ 1\ 2\ 3)$.

We wish to test $\mathbf{H}_0'\boldsymbol{\beta} = 0$, where

$$\mathbf{H}_0' = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$$

against $\mathbf{H}_a'\boldsymbol{\beta} = 0$, where $\mathbf{H}_a' = [1\ \text{-}1\ \text{-}1\ 1]$. Note that (-1 1) $\mathbf{H}_0' = \mathbf{H}_a'$ and both are estimable. Therefore these are valid hypotheses. Using the reduction method we solve

$$\begin{pmatrix} 6 & 3 & 2 & 1 & 1 & 3 \\ 3 & 7 & 1 & 2 & -1 & 1 \\ 2 & 1 & 8 & 1 & -1 & 2 \\ 1 & 2 & 1 & 9 & 1 & 3 \\ 1 & -1 & -1 & 1 & 0 & 0 \\ 3 & 1 & 2 & 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}_a \\ \boldsymbol{\theta}_a \end{pmatrix} = \begin{pmatrix} 9 \\ 12 \\ 15 \\ 16 \\ 0 \\ 0 \end{pmatrix}.$$

The solution is [-1876, 795, -636, 2035, -20035, 20310]/3643, and the reduction under $\mathbf{H}_a'$ is 15676/3643 = 4.3030. Then solve

$$\begin{pmatrix} 6 & 3 & 2 & 1 & 1 & 2 & 3 \\ 3 & 7 & 1 & 2 & 2 & 1 & 1 \\ 2 & 1 & 8 & 1 & 1 & 0 & 2 \\ 1 & 2 & 1 & 9 & 0 & 1 & 3 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 3 & 1 & 2 & 3 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}_0 \\ \boldsymbol{\theta}_0 \end{pmatrix} = \begin{pmatrix} 9 \\ 12 \\ 15 \\ 16 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The solution is [-348, 290, -232, 406, 4380, -5302, 5088]/836, and the reduction is 3364/836 = 4.0239. Then we test 4.3030 - 4.0239 = .2791 entering $\chi^2$ with 1 degree of freedom coming from the differences between the number of rows in $\mathbf{H}_0'$ and $\mathbf{H}_a'$.

By the method involving $Var(\mathbf{H}_o'\boldsymbol{\beta})$ and $Var(\mathbf{H}_a'\boldsymbol{\beta})$ we solve the following equations and find a g-inverse of the coefficient matrix.

$$\begin{pmatrix} 6 & 3 & 2 & 1 & 3 \\ 3 & 7 & 1 & 2 & 1 \\ 2 & 1 & 8 & 1 & 2 \\ 1 & 2 & 1 & 9 & 3 \\ 3 & 1 & 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}^o \\ \boldsymbol{\theta}_0 \end{pmatrix} = \begin{pmatrix} 9 \\ 12 \\ 15 \\ 16 \\ 0 \end{pmatrix}.$$

The solution is [-7664, 8075, 5561, 1265, 18040]/4972. The inverse is

$$\begin{pmatrix} 624 & -276 & -336 & -308 & 1012 \\ & 887 & 53 & -55 & -352 \\ & & 659 & -121 & 220 \\ & & & 407 & 616 \\ & & & & -2024 \end{pmatrix} /4972.$$

Now

$$\mathbf{H}_0'\boldsymbol{\beta}^o = [2.82522 \;\; -1.20434]',$$

$$\mathbf{H}_0'\mathbf{C}_{11}\mathbf{H}_0 \;\; = \;\; \begin{pmatrix} .65708 & .09735 \\ & .27031 \end{pmatrix},$$

and

$$(\mathbf{H}_0'\mathbf{C}_{11}\mathbf{H}_0)^{-1} \;\; = \;\; \begin{pmatrix} 1.60766 & -.57895 \\ & 3.90789 \end{pmatrix} = \;\mathbf{B},$$

where $\mathbf{C}_{11}$ is the upper $4 \times 4$ submatrix of the inverse of the coefficient matrix. Then

$$[2.82522 \;\; -1.20434] \; \mathbf{B} \; [2.8522 \;\; -1.20434]' \;\; = \;\; 22.44007.$$

Similarly computations with $\mathbf{H}_a' = (1\ \text{-1}\ \text{-1}\ 1)$, give $\mathbf{H}_a'\boldsymbol{\beta}_a$ = -4.02957, $\mathbf{B}$ = 1.36481, and $(-4.02957)\mathbf{B}(-4.02957) = 22.16095$. Then 22.44007 - 22.16095 = .2791 as before.