Chapter 6
G and R Known to Proportionality

C. R. Henderson

1984 - Guelph

In the preceding chapters it has been assumed that $Var(\mathbf{u}) = \mathbf{G}$ and $Var(\mathbf{e}) = \mathbf{R}$ are known. This is, of course, an unrealistic assumption, but was made in order to present estimation, prediction, and hypothesis testing methods that are exact and which may suggest approximations for the situation with unknown $\mathbf{G}$ and $\mathbf{R}$. One case does exist, however, in which BLUE and BLUP exist, and exact tests can be made even when these variances are unknown. This case is $\mathbf{G}$ and $\mathbf{R}$ known to proportionality.

Suppose that we know $\mathbf{G}$ and $\mathbf{R}$ to proportionality, that is

$$\begin{aligned} \mathbf{G} &= \mathbf{G}_* \ \sigma_e^2, \\ \mathbf{R} &= \mathbf{R}_* \ \sigma_e^2. \end{aligned} \tag{1}$$

$\mathbf{G}_*$ and $\mathbf{R}_*$ are known, but σ_e^2 is not. For example, suppose that we have a one way mixed model

$$\begin{aligned} y_{ij} &= \mathbf{x}_{ij}'\boldsymbol{\beta} + a_i + e_{ij}. \\ Var(a_1 \ \ a_2 \ ...)' &= \mathbf{I}\sigma_a^2. \\ Var(e_{11} \ \ e_{12} \ ...)' &= \mathbf{I}\sigma_e^2. \end{aligned}$$

Suppose we know that $\sigma_a^2/\sigma_e^2 = \alpha$. Then

$$\begin{aligned} \mathbf{G} &= \mathbf{I}\sigma_a^2 = \mathbf{I}\alpha \ \sigma_e^2. \\ \mathbf{R} &= \mathbf{I}\sigma_e^2. \end{aligned}$$

Then by the notation of (1)

$$\mathbf{G}_* = \mathbf{I}\alpha, \ \mathbf{R}_* = \mathbf{I}.$$

1 BLUE and BLUP

Let us write the GLS equations with the notation of (1).

$$\begin{aligned} \mathbf{V} &= \mathbf{ZGZ}' + \mathbf{R} \\ &= (\mathbf{ZG}_*\mathbf{Z}' + \mathbf{R}_*)\sigma_e^2 \\ &= \mathbf{V}_*\sigma_e^2. \end{aligned}$$

Then $\mathbf{X}'\mathbf{V}^{-1}\mathbf{X}\boldsymbol{\beta}^o = \mathbf{X}'\mathbf{V}^{-1}\mathbf{y}$ can be written as

$$\sigma_e^{-2}\mathbf{X}'\mathbf{V}_*^{-1}\mathbf{X}\boldsymbol{\beta}^o = \mathbf{X}'\mathbf{V}_*^{-1}\mathbf{y}\sigma_e^{-2}. \tag{2}$$

Multiplying both sides by σ_e^2 we obtain a set of equations that can be written as,

$$\mathbf{X}'\mathbf{V}_*^{-1}\mathbf{X}\boldsymbol{\beta}^o = \mathbf{X}'\mathbf{V}_*^{-1}\mathbf{y}. \tag{3}$$

Then BLUE of $\mathbf{K}'\boldsymbol{\beta}$ is $\mathbf{K}'\boldsymbol{\beta}^o$, where $\boldsymbol{\beta}^o$ is any solution to (3).

Similarly the mixed model equations with each side multiplied by σ_e^2 are

$$\begin{pmatrix} \mathbf{X}'\mathbf{R}_*^{-1}\mathbf{X} & \mathbf{X}'\mathbf{R}_*^{-1}\mathbf{Z} \\ \mathbf{Z}'\mathbf{R}_*^{-1}\mathbf{X} & \mathbf{Z}'\mathbf{R}_*^{-1}\mathbf{Z} + \mathbf{G}_*^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}^o \\ \hat{\mathbf{u}} \end{pmatrix} = \begin{pmatrix} \mathbf{X}'\mathbf{R}_*^{-1}\mathbf{y} \\ \mathbf{Z}'\mathbf{R}_*^{-1}\mathbf{y} \end{pmatrix}. \tag{4}$$

$\hat{\mathbf{u}}$ is BLUP of $\mathbf{u}$ when $\mathbf{G}_*$ and $\mathbf{R}_*$ are known.

To find the sampling variance of $\mathbf{K}'\boldsymbol{\beta}^o$ we need a g-inverse of the matrix of (2). This is

$$(\mathbf{X}'\mathbf{V}_*^{-1}\mathbf{X})^- \ \sigma_e^2.$$

Consequently,

$$Var(\mathbf{K}'\boldsymbol{\beta}^o) = \mathbf{K}'(\mathbf{X}'\mathbf{V}_*^{-1}\mathbf{X})^-\mathbf{K}\sigma_e^2. \tag{5}$$

Also

$$Var(\mathbf{K}'\boldsymbol{\beta}^o) = \mathbf{K}'\mathbf{C}_{11}\mathbf{K}\sigma_e^2,$$

where $\mathbf{C}_{11}$ is the upper p^2 submatrix of a g-inverse of the matrix of (4). Similarly all of the results of (34) to (41) in Chapter 5 are correct if we multiply them by σ_e^2.

Of course σ_e^2 is unknown, so we can only estimate the variance by substituting some estimate of σ_e^2, say $\hat{\sigma}_e^2$, in (5). There are several methods for estimating σ_e^2, but the most frequently used one is the minimum variance, translation invariant, quadratic, unbiased estimator computed by

$$[\mathbf{y}'\mathbf{V}_*^{-1}\mathbf{y} - (\boldsymbol{\beta}^o)'\mathbf{X}'\mathbf{V}_*^{-1}\mathbf{y}]/[n - rank(\mathbf{X})] \tag{6}$$

or by

$$[\mathbf{y}'\mathbf{R}_*^{-1}\mathbf{y} - (\boldsymbol{\beta}^o)'\mathbf{X}'\mathbf{R}_*^{-1}\mathbf{y} - \hat{\mathbf{u}}'\mathbf{Z}'\mathbf{R}_*^{-1}\mathbf{y}]/[n - rank(\mathbf{X})]. \tag{7}$$

A more detailed account of estimation of variances is presented in Chapters 10, 11, and 12.

Next looking at BLUP of $\mathbf{u}$ under model (1), it is readily seen that $\hat{\mathbf{u}}$ of (4) is BLUP. Similarly variances and covariances involving $\hat{\mathbf{u}}$ and $\hat{\mathbf{u}} - \mathbf{u}$ are easily derived from the results for known $\mathbf{G}$ and $\mathbf{R}$. Let

$$\begin{pmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{12} & \mathbf{C}_{22} \end{pmatrix}$$

be a g-inverse of the matrix of (4). Then

$$
\begin{aligned}
Cov(\mathbf{K}'\boldsymbol{\beta}^o, \hat{\mathbf{u}}' - \mathbf{u}') &= \mathbf{K}'\mathbf{C}_{12}\sigma_e^2, \qquad (8)\\
Var(\hat{\mathbf{u}}) &= (\mathbf{G}_* - \mathbf{C}_{22})\sigma_e^2, \qquad (9)\\
Var(\hat{\mathbf{u}} - \mathbf{u}) &= \mathbf{C}_{22}\sigma_e^2. \qquad (10)
\end{aligned}
$$

2 Tests of Hypotheses

In the same way in which $\mathbf{G}$ and $\mathbf{R}$ known to proportionality pose no problems in BLUE and BLUP, exact tests of hypotheses regarding $\boldsymbol{\beta}$ can be performed, assuming as before a multivariate normal distribution. Chapter 4 describes computation of a quadratic, s, that is distributed as χ^2 with $m - a$ degrees of freedom when the null hypothesis is true, and m and a are the number of rows in $\mathbf{H}_0'$ and $\mathbf{H}_a'$ respectively. Now we compute these quadratics exactly as in these methods except that $\mathbf{V}_*$, $\mathbf{G}_*$, $\mathbf{R}_*$ are substituted for $\mathbf{V}$, $\mathbf{G}$, $\mathbf{R}$. Then when the null hypothesis is true, $s/\hat{\sigma}_e^2(m-a)$ is distributed as F with $m - a$, and $n-$rank $(\mathbf{X})$ degrees of freedom, where $\hat{\sigma}_e^2$ is computed by (6) or (7).

3 Power Of The Test Of Null Hypotheses

Two different types of errors can be made in tests of hypotheses. First, the null hypothesis may be rejected when in fact it is true. This is commonly called a Type 1 error. Second, the null hypothesis may be accepted when it is really not true. This is called a Type 2 error, and the power of the test is defined as 1 minus the probability of a Type 2 error. The results that follow regarding power assume that $\mathbf{G}_*$ and $\mathbf{R}_*$ are known.

The power of the test can be computed only if

1. The true value of $\boldsymbol{\beta}$ for which the power is to be determined is specified. Different values of $\boldsymbol{\beta}$ give different powers. Let this value be $\boldsymbol{\beta}_t$. Of course we do not know the true value, but we may be interested in the power of the test, usually for some minimum differences among elements of $\boldsymbol{\beta}$. Logically $\boldsymbol{\beta}_t$ must be true if the null and the alternative hypotheses are true. Accordingly a $\boldsymbol{\beta}_t$ must be chosen that violates neither $\mathbf{H}_0'\boldsymbol{\beta} = \mathbf{c}_0$ nor $\mathbf{H}_a'\boldsymbol{\beta} = \mathbf{c}_a$.

2. The probability of the type 1 error must be specified. This is often called the chosen significance level of the test.

3. The value of $\hat{\sigma}_e^2$ must be specified. Because the power should normally be computed prior to the experiment, this would come from prior research. Define this value as d.

4. **X** and **Z** must be specified.

Then let

$$\begin{aligned} \text{A} &= \text{significance level} \\ F_1 &= m - a \;=\; \text{numerator d.f.} \\ F_2 &= n - \text{rank}(\mathbf{X}) \;=\; \text{denominator d.f.} \end{aligned}$$

Compute $\triangle$ = the quadratic, s, but with $\mathbf{X}\boldsymbol{\beta}_t$ substituted for $\mathbf{y}$ in the computations. Compute

$$P \;=\; [\triangle/(m-a+1)d]^{1/2} \tag{11}$$

and enter Tiku's table (1967) with $A,\ F_1,\ F_2,\ P$ to find the power of the test.

Let us illustrate computation of power by a simple one-way fixed model,

$$\begin{aligned} y_{ij} &= \mu + t_i + e_{ij}, \\ i &= 1, 2, 3. \\ Var(\mathbf{e}) &= \mathbf{I}\sigma_e^2. \end{aligned}$$

Suppose there are 3,2,4 observations respectively on the 3 treatments. We wish to test

$$\mathbf{H}_0'\boldsymbol{\beta} \;=\; \mathbf{0},$$

where

$$\mathbf{H}_0' \;=\; \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix},$$

against the unrestricted hypothesis.

Suppose we want the power of the test for $\boldsymbol{\beta}_t' \;=\; [10, 2, 1, -3]$ and $\sigma_e^2 \;=\; 12$. That is, $d = 12$. Then

$$(\mathbf{X}\boldsymbol{\beta}_t)' \;=\; [12,\ 12,\ 12,\ 11,\ 11,\ 7,\ 7,\ 7,\ 7].$$

As we have shown, the reduction under the null hypothesis in this case can be found from the reduced model $E(\mathbf{y}) \;=\; \mu_0$. The OLS equations are

$$\begin{pmatrix} 9 & 3 & 2 & 4 \\ 3 & 3 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 4 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} \mu^o \\ t^o \end{pmatrix} \;=\; \begin{pmatrix} 86 \\ 36 \\ 22 \\ 28 \end{pmatrix}.$$

A solution is (0, 12, 11, 7), and reduction = 870. The restricted equations are

$$9\ \mu^o \;=\; 86,$$

and the reduction is 821.78. Then $s = 48.22 = \triangle$. Let us choose $A = .05$ as the significance level

$$\begin{aligned} F_1 &= 2 - 0 = 2. \\ F_2 &= 9 - 3 = 6. \\ P &= \frac{48.22}{3(12)} = 1.157. \end{aligned}$$

Entering Tiku's table we obtain the power of the test.