

# Chapter 30

## Line Cross and Breed Cross Analyses

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This chapter is concerned with a genetic model for line crosses, BLUP of crosses, and estimation of variances. It is assumed that a set of unselected inbred lines is derived from some base population. Therefore the lines are assumed to be uncorrelated.

### 1 Genetic Model

We make the assumption that the total genetic variance of a population can be partitioned into additive + dominance + (additive  $\times$  additive) + (additive  $\times$  dominance), etc. Further, in a non-inbred population these different sets of effects are mutually uncorrelated, e.g.,  $\text{Cov}(\text{additive}, \text{dominance}) = \mathbf{0}$ . The covariance among sets of effects can be computed from the  $\mathbf{A}$  matrix. Methods for computing  $\mathbf{A}$  are well known.  $\mathbf{D}$  can be computed as described in Chapter 29.

$$\text{Var}(\text{additive effects}) = \mathbf{A}\sigma_a^2.$$

$$\text{Var}(\text{dominance effects}) = \mathbf{D}\sigma_d^2.$$

$$\text{Var}(\text{additive} \times \text{dominance}) = \mathbf{A}\#\mathbf{D}\sigma_{ad}^2.$$

$$\text{Var}(\text{additive} \times \text{additive} \times \text{dominance}) = \mathbf{A}\#\mathbf{A}\#\mathbf{D}\sigma_{aad}^2, \text{ etc.}$$

$\#$  denotes the operation of taking the product of corresponding elements of 2 matrices. Thus the  $ij^{\text{th}}$  element of  $\mathbf{A}\#\mathbf{D}$  is  $a_{ij}d_{ij}$ .

### 2 Covariances Between Crosses

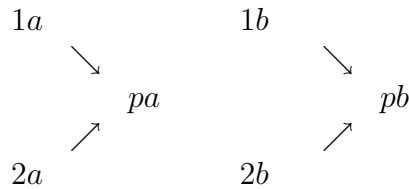
If lines are unrelated, the progeny resulting from line crosses are non-inbred and consequently the covariance matrices for the different genetic components can be computed for the progeny. Then one can calculate BLUP for these individual animals by the method described in Chapter 29. With animals as contrasted to plants it would seem wise to include a maternal influence of line of dam in the model as described below. Now in order to reduce computational labor we shall make some simplifying assumptions as follows.

1. All members of all lines have inbreeding coefficient =  $f$ .
2. The lines are large enough that two random individuals from the same line are unrelated except for the fact that they are members of the same line.

Consequently, the  $\mathbf{A}$  matrix for members of the same line is

$$\begin{pmatrix} 1+f & & 2f \\ & \ddots & \\ 2f & & 1+f \end{pmatrix}.$$

From this result we can calculate the covariance between any random pair of individuals from the same cross or a random individual of one cross with a random individual of another cross. We illustrate first with single crosses. Consider line cross,  $1 \times 2$ , line 1 being used as the sire line. Two random progeny pedigrees can be visualized as



Therefore

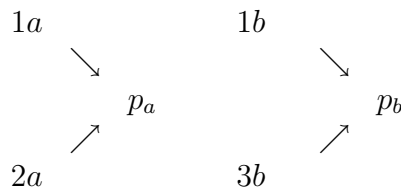
$$\begin{aligned} a_{1a,1b} &= a_{2a,2b} = 2f. \\ a_{pa,pb} &= .25(2f + 2f) = f. \\ d_{pa,pb} &= .25[2f(2f) + 0(0)] = f^2. \end{aligned}$$

Then the genetic covariance between 2 random members of any single cross is equal to the genetic variance of single cross means

$$= f\sigma_a^2 + f^2\sigma_d^2 + f^2\sigma_{aa}^2 + f^3\sigma_{ad}^2 + \text{etc.}$$

Note that if  $f = 1$ , this simplifies to the total genetic variance of individuals in the population from which the lines were derived.

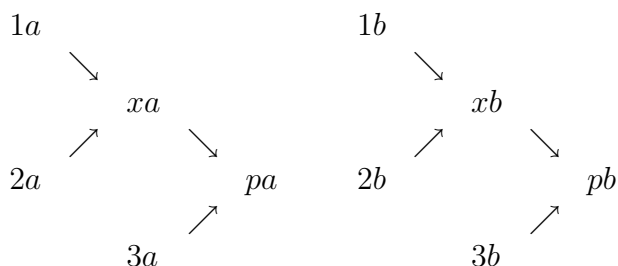
Next consider the covariance between crosses with one common parental line, say  $1 \times 2$  with  $1 \times 3$ .



As before,  $a_{1a,1b} = 2f$ , but all other relationships among parental pairs are zero. Then

$$\begin{aligned} a_{pa,pb} &= .25(2f) = .5f. \\ d_{pa,pb} &= 0. \\ \text{Covariance} &= .5f\sigma_a^2 + .25f^2\sigma_{aa}^2 + \dots, \text{etc.} \end{aligned}$$

Next we consider 3 way crosses. Represent 2 random members of a 3 way cross ( $1 \times 2$ )  $\times$  3 by



Non-zero additive relationships are

$$\begin{aligned} (1a, 1b) &= (2a, 2b) = (3a, 3b) = 2f, \text{ and} \\ (xa, xb) &= f, \\ (pa, pb) &= .25(f + 2f) = .75f, \end{aligned}$$

and the dominance relationship is

$$(pa, pb) = .25[f(2f) + 0(0)] = .5f^2.$$

Thus the genetic variance of 3 way crosses is

$$\frac{3}{4} f\sigma_a^2 + \frac{1}{2} f^2\sigma_d^2 + \frac{3}{8} f^3\sigma_{ad}^2 + \frac{9}{16} f^2\sigma_{aa}^2 + \dots \text{etc.}$$

The covariance between a single cross and a 3 way cross depends upon the way the crosses are made.

For a  $(1 \times 2) \times 3$  with  $1 \times 2$  is  $f/2$ , and  $d$  is 0.

For a  $(1 \times 2) \times 3$  with  $1 \times 3$  is  $.75f$ , and  $d$  is  $.5f^2$ .

The variance of 4 way crosses is  $.5 f\sigma_a^2 + .25 f^2\sigma_d^2 + \dots$  etc. The variance of top crosses with an inbred line as a parent is  $.5 f\sigma_a^2 + (0)\sigma_d^2 + \text{etc.}$

If we know the magnitude of the various components of genetic variance, we can derive the variance of any line cross or the covariance between any pair of line crosses. Then these can be used to set up mixed model equations. One must be alert to the possibility that some of the variance-covariance matrices of genetic components may be singular.

### 3 Reciprocal Crosses Assumed Equal

This section is concerned with a model in which the cross, line  $i \times$  line  $j$ , is assumed the same as the cross, line  $j \times$  line  $i$ . The model is

$$y_{ijk} = \mathbf{x}'_{ijk}\boldsymbol{\beta} + c_{ij} + e_{ijk}.$$

$\text{Var}(\mathbf{c})$  has this form

$$\begin{aligned} \text{Var}(c_{ij}) &= f\sigma_a^2 + f^2\sigma_d^2 + f^3\sigma_{ad}^2 + \text{etc.} \\ &= \text{Cov}(c_{ij}, c_{ji}). \\ \text{Cov}(c_{ij}, c_{ij'}) &= \text{Cov}(c_{ij}, c_{ji'}) = .5 f\sigma_a^2 + .25 f^2\sigma_{aa}^2 + \dots \text{etc.} \\ \text{Cov}(c_{ij}, c_{i'j'}) &= 0. \end{aligned}$$

We illustrate BLUP with single crosses among 4 lines with  $f = .6$ ,  $\sigma_a^2 = .4$ ,  $\sigma_d^2 = .3$ ,  $\sigma_e^2 = 1$ . All other genetic covariances are ignored.  $\boldsymbol{\beta} = \boldsymbol{\mu}$ . The number of observations per cross and  $\bar{y}_{ij}$ , are

$n_{ij}$				$\bar{y}_{ij}$			
X	5	3	2	X	6	4	7
4	X	6	3	5	X	3	8
4	2	X	5	6	7	X	3
2	3	9	X	5	6	4	X

X denotes no observation. The OLS equations are in (30.1). Note that  $a_{ij}$  is combined with  $a_{ji}$  to form the variable  $a_{ij}$  and similarly for  $\mathbf{d}$ .

$$\begin{pmatrix} 48 & 9 & 7 & 4 & 8 & 6 & 14 & 9 & 7 & 4 & 8 & 6 & 14 \\ & 9 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 \\ & & 7 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 \\ & & & 4 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ & & & & 8 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\ & & & & & 6 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ & & & & & & 14 & 0 & 0 & 0 & 0 & 0 & 14 \\ & & & & & & & 9 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & & 7 & 0 & 0 & 0 & 0 \\ & & & & & & & & & 4 & 0 & 0 & 0 \\ & & & & & & & & & & 8 & 0 & 0 \\ & & & & & & & & & & & 6 & 0 \\ & & & & & & & & & & & & 14 \end{pmatrix} \begin{pmatrix} \mu \\ a_{12} \\ a_{13} \\ a_{14} \\ a_{23} \\ a_{24} \\ a_{34} \\ d_{12} \\ d_{13} \\ d_{14} \\ d_{23} \\ d_{24} \\ d_{34} \end{pmatrix} = \begin{pmatrix} 235 \\ 50 \\ 36 \\ 24 \\ 32 \\ 42 \\ 51 \\ 50 \\ 36 \\ 24 \\ 32 \\ 42 \\ 51 \end{pmatrix} \quad (1)$$

$$Var(\mathbf{a}) = \begin{pmatrix} .24 & .12 & .12 & .12 & .12 & 0 \\ & .24 & .12 & .12 & 0 & .12 \\ & & .24 & 0 & .12 & .12 \\ & & & .24 & .12 & .12 \\ & & & & .24 & .12 \\ & & & & & .24 \end{pmatrix}, \quad Var(\mathbf{d}) = .108 \mathbf{I}.$$

$Var(\mathbf{a})$  is singular. Consequently we pre-multiply equation (30.1) by

$$\begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Var(\mathbf{a}) & \mathbf{0} \\ \mathbf{0} & 0 & \mathbf{I} \end{pmatrix}$$

and add

$$\begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & [Var(\mathbf{d})]^{-1} \end{pmatrix}$$

to the resulting coefficient matrix. The solution to these equations is

$$\begin{aligned} \hat{\mu} &= 5.1100, \\ \hat{\mathbf{a}}' &= (.5528, -.4229, .4702, -.4702, .4229, -.5528), \\ \hat{\mathbf{d}}' &= (-.0528, .1962, .1266, -.2965, .5769, -.5504). \end{aligned}$$

Note that  $\sum \hat{d} = 0$ . Now the predicted future progeny average of the  $ij^{th}$  and  $ji^{th}$  cross is

$$\mu^* + \hat{a}_{ij} + \hat{d}_{ij},$$

where  $\mu^*$  is the fixed part of the model for future progeny.

If we want to predict the future progeny mean of a cross between  $i \times k$  or between  $k \times i$ , where  $k$  is not in the sample, we can do this by selection index methods using  $\hat{\mathbf{a}}, \hat{\mathbf{d}}$  as the “data” with variances and covariances applying to  $\mathbf{a} + \mathbf{d}$  rather than  $\mathbf{a}$ . See Section 5.9. For example the prediction of the  $1 \times 5$  cross is

$$(.12 \ .12 \ .12 \ 0 \ 0 \ 0) \begin{pmatrix} .348 & .12 & .12 & .12 & .12 & 0 \\ & .348 & .12 & .12 & 0 & .12 \\ & & .348 & 0 & .12 & .12 \\ & & & .348 & .12 & .12 \\ & & & & .348 & .12 \\ & & & & & .348 \end{pmatrix}^{-1} (\hat{\mathbf{a}} + \hat{\mathbf{d}}). \quad (2)$$

If we were interested only in prediction of crosses among the lines 1 2, 3, 4, we could reduce the mixed model equations to solve for  $\hat{\mathbf{a}} + \hat{\mathbf{d}}$  jointly. Then there would be only 7 equations. The  $6 \times 6$  matrix of (30.2) would be  $\mathbf{G}^{-1}$  to add to the lower  $6 \times 6$  submatrix of the least squares coefficient matrix.

## 4 Reciprocal Crosses With Maternal Effects

In most animal breeding models one would assume that because of maternal effects the  $ij^{th}$  cross would be different from the  $ji^{th}$ . Now the genetic model for maternal effects involves the genetic merit with respect to maternal of the female line in a single cross. This complicates statements of the variances and covariances contributed by different genetic components since the lines are inbred. The statement of  $\sigma_a^2$  is possible but not the others. The contribution of  $\sigma_a^2$  is

$$\text{Covariance between 2 progeny of the same cross} = 2f\sigma_a^2,$$

$$\text{Covariance between progeny of } i \times j \text{ with } k \times j = .5f\sigma_a^2,$$

where the second subscript denotes the female line. Consequently if we ignore other components, we need only to add  $m_j$  to the model with  $Var(\mathbf{m}) = \mathbf{I}\sigma_m^2$ . We illustrate with the same data as in Section 30.3 with  $Var(\mathbf{m}) = .5\mathbf{I}$ . The OLS equations now are in (30.3). Now we pre-multiply these equations by

$$\begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Var(\mathbf{a}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix}.$$

Then add to the resulting coefficient matrix

$$\begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & [Var(\mathbf{d})]^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & [Var(\mathbf{m})]^{-1} \end{pmatrix}.$$

The resulting solution is

$$\begin{aligned} \hat{\mu} &= 5.1999, \\ \hat{\mathbf{a}}' &= (.2988, -.2413, .3217, -.3217, .2413, -.2988), \\ \hat{\mathbf{d}}' &= (-.1737, .2307, .1136, -.1759, .4479, -.4426), \end{aligned}$$

and

$$\hat{\mathbf{m}}' = (.0560, .6920, -.8954, .1475).$$

$$\begin{pmatrix}
48 & 9 & 7 & 4 & 8 & 6 & 14 & 9 & 7 & 4 & 8 & 6 & 14 & 10 & 10 & 18 & 10 \\
& 9 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 & 4 & 5 & 0 & 0 \\
& & 7 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 4 & 0 & 3 & 0 \\
& & & 4 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 2 & 0 & 0 & 2 \\
& & & & 8 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 2 & 6 & 0 \\
& & & & & 6 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 3 & 0 & 3 \\
& & & & & & 14 & 0 & 0 & 0 & 0 & 0 & 14 & 0 & 0 & 9 & 5 \\
& & & & & & & 9 & 0 & 0 & 0 & 0 & 0 & 4 & 5 & 0 & 0 \\
& & & & & & & & 7 & 0 & 0 & 0 & 0 & 4 & 0 & 3 & 0 \\
& & & & & & & & & 4 & 0 & 0 & 0 & 2 & 0 & 0 & 2 \\
& & & & & & & & & & 8 & 0 & 0 & 0 & 2 & 6 & 0 \\
& & & & & & & & & & & 6 & 0 & 0 & 3 & 0 & 3 \\
& & & & & & & & & & & & 14 & 0 & 0 & 9 & 5 \\
& & & & & & & & & & & & & 10 & 0 & 0 & 0 \\
& & & & & & & & & & & & & & 10 & 0 & 0 \\
& & & & & & & & & & & & & & & 18 & 0 \\
& & & & & & & & & & & & & & & & 10
\end{pmatrix}
\begin{pmatrix}
\hat{\mu} \\
\mathbf{a} \\
\mathbf{d} \\
\mathbf{m}
\end{pmatrix}
= (235, 50, 36, 24, 32, 42, 51, 50, 36, 24, 32, 42, 51, 54, 62, 66, 53)'. \quad (3)$$

## 5 Single Crosses As The Maternal Parent

If single crosses are used as the maternal parent in crossing, we can utilize various components of genetic variation with respect to maternal effects, for then the maternal parents are non-inbred.

## 6 Breed Crosses

If one set of breeds is used as males and a second different set is used as females in a breed cross, the problem is the same as for any two way fixed cross-classified design with interaction and possible missing subclasses. If there is no missing subclass, the weighted squares of means analysis would seem appropriate, but with small numbers of progeny per cross,  $\bar{y}_{ij}$  may not be the optimum criterion for choosing the best cross. Rather, we might choose to treat the interaction vector as a pseudo-random variable and proceed to a biased estimation that might well have smaller mean squared error than the  $\bar{y}_{ij}$ . If subclasses are missing, this biased procedure enables finding a biased estimator of such crosses.

## 7 Same Breeds Used As Sires And Dams

If the same breeds are used as sires and as dams and with progeny of some or all of the pure breeds included in the design, the analysis can be more complicated. Again one possibility is to evaluate a cross or pure line simply by the subclass mean. However, most breeders have attempted a more complicated analysis involving, for example, the following model for  $\mu_{ij}$  the true mean of the cross between the  $i^{th}$  sire breed and the  $j^{th}$  dam breed.

$$\begin{aligned}\mu_{ij} &= \mu + s_i + d_j + \gamma_{ij} + p \text{ if } i = j \\ &= \mu + s_i + d_j + \gamma_{ij} \text{ if } i \neq j.\end{aligned}$$

From the standpoint of ranking crosses by BLUE, this model is of no particular value, for even with filled subclasses the rank of the coefficient matrix is only  $b^2$ , where  $b$  is the number of breeds. A solution to the OLS equations is

$$\begin{aligned}\mu^o &= \mathbf{s}^o = \mathbf{d}^o = p^o = \mathbf{0} \\ \hat{\gamma}_{ij} &= \bar{y}_{ij}.\end{aligned}$$

Thus BLUE of a breed cross is simply  $\bar{y}_{ij}$ , provided  $n_{ij} > 0$ . The extended model provides no estimate of a missing cross since that is not estimable. In contrast, if one is prepared to use biased estimation, a variety of estimates of missing crosses can be derived, and these same biased estimators may, in fact, be better estimators of filled subclasses than  $\bar{y}_{ij}$ . Let us restrict ourselves to estimators of  $\mu_{ij}$  that have expectation,  $\mu + s_i + d_j + p$  + linear function of  $\gamma$  if  $i = j$ , or  $\mu + s_i + d_j +$  linear function of  $\gamma$  if  $i \neq j$ . Assume that the  $\gamma_{ii}$  are different from the  $\gamma_{ij}$  ( $i \neq j$ ). Accordingly, let us assume for convenience that

$$\begin{aligned}\sum_{j=1}^b \gamma_{ij} &= 0 \text{ for } i = 1, \dots, b, \\ \sum_{i=1}^b \gamma_{ij} &= 0 \text{ for } j = 1, \dots, b, \text{ and} \\ \sum_{i=1}^b \gamma_{ii} &= 0.\end{aligned}$$

Next permute all labelling of breeds and compute the average squares and products of the  $\gamma_{ij}$ . These have the following form:

$$\begin{aligned}\text{Av.}(\gamma_{ii})^2 &= d. \\ \text{Av.}(\gamma_{ij})^2 &= c. \\ \text{Av.}(\gamma_{ii} \gamma_{jj}) &= -d/(b-1).\end{aligned}$$



$$\begin{aligned}
\text{Av.}(\gamma_{ij} \gamma_{ik}) &= \text{Av.}(\gamma_{ij} \gamma_{kj}) = \frac{d - c(b - 1)}{(b - 1)(b - 2)}. \\
\text{Av.}(\gamma_{ii} \gamma_{ij}) &= -d/(b - 1). \\
\text{Av.}(\gamma_{ii} \gamma_{ji}) &= -d/(b - 1). \\
\text{Av.}(\gamma_{ij} \gamma_{ji}) &= r. \\
\text{Av.}(\gamma_{ii} \gamma_{jk}) &= 2d/(b - 1)(b - 2). \\
\text{Av.}(\gamma_{ij} \gamma_{ki}) &= \frac{d - r(b - 1)}{(b - 1)(b - 2)}. \\
\text{Av.}(\gamma_{ij} \gamma_{kl}) &= \frac{(c + r)(b - 1) - 4d}{(b - 1)(b - 2)(b - 3)}. \\
\text{Av.}(\gamma_{ij} \gamma_{jk}) &= \text{Av.}(\gamma_{ij} \gamma_{ki}).
\end{aligned}$$

These squares and products comprise a singular  $\mathbf{P}$  matrix which could then be used in pseudo-mixed model equations. This would, of course, require estimating  $d$ ,  $c$ ,  $r$  from the data. Solving the resulting mixed model type equations,

$$\begin{aligned}
\hat{\mu}_{ii} &= \mu^o + s_i^o + d_i^o + \hat{\gamma}_{ii} + p^o, \\
\hat{\mu}_{ij} &= \mu^o + s_i^o + d_i^o + \hat{\gamma}_{ij},
\end{aligned}$$

when  $i \neq j$ .

A simpler method is to pretend that the model for  $\mu_{ij}$  is

$$\mu_{ij} = \mu + s_i + d_j + \gamma_{ij} + r_{(i,j)},$$

when  $i \neq j$ , and

$$\mu_{ii} = \mu + s_i + d_j + \gamma_{ii} + p.$$

$\mathbf{r}$  has  $b(b - 1)/2$  elements and  $(ij)$  denotes  $i < j$ . Thus the element of  $\mathbf{r}$  for  $\mu_{ij}$  is the same as for  $\mu_{ji}$ . Then partition  $\boldsymbol{\gamma}$  into the  $\gamma_{ii}$  elements and the  $\gamma_{ij}$  elements and pretend that  $\boldsymbol{\gamma}$  and  $\mathbf{r}$  are random variables with

$$\text{Var} \begin{pmatrix} \gamma_{11} \\ \gamma_{22} \\ \vdots \end{pmatrix} = \mathbf{I}\sigma_1^2, \text{Var} \begin{pmatrix} \gamma_{12} \\ \gamma_{13} \\ \vdots \end{pmatrix} = \mathbf{I}\sigma_2^2, \text{Var}(\mathbf{r}) = \mathbf{I}\sigma_3^2.$$

The covariances between these three vectors are all null. Then set up and solve the mixed model equations. With proper choices of values of  $\sigma_1^2$ ,  $\sigma_2^2$ ,  $\sigma_3^2$  relative to  $b$ ,  $d$ ,  $c$ ,  $r$  the estimates of the breed crosses are identical to the previous method using singular  $\mathbf{P}$ . The latter method is easier to compute and it is also much easier to estimate  $\sigma_1^2$ ,  $\sigma_2^2$ ,  $\sigma_3^2$  than the parameters of  $\mathbf{P}$ . For example, we could use Method 3 by computing appropriate reductions and equating to their expectations.

We illustrate these two methods with a 4 breed cross. The  $n_{ij}$  and  $y_{ij}$  were as follows.

$n_{ij}$				$y_{ij}$			
5	2	3	1	6	3	2	7
4	2	6	7	8	3	5	6
3	5	2	8	9	4	7	3
4	2	3	4	2	6	8	6

Assume that  $\mathbf{P}$  is the following matrix, (30.4) ... (30.6).  $Var(\mathbf{e}) = \mathbf{I}$ . Then we premultiply the OLS equations by

$$\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{P} \end{pmatrix}$$

and add  $\mathbf{I}$  to the last 16 diagonal coefficients.

Upper  $8 \times 8$

$$\begin{pmatrix} 1.8 & -.6 & -.6 & -.6 & -.6 & -.6 & .6 & .6 \\ & 4.48 & -1.94 & -1.94 & .88 & -.6 & -.14 & -.14 \\ & & 4.48 & -1.94 & -.14 & .6 & -1.94 & 1.48 \\ & & & 4.48 & -.14 & .6 & 1.48 & -1.94 \\ & & & & 4.48 & -.6 & -1.94 & -1.94 \\ & & & & & 1.8 & -.6 & -.6 \\ & & & & & & 4.48 & -1.94 \\ & & & & & & & 4.48 \end{pmatrix} \quad (4)$$

Upper right  $8 \times 8$  and (lower left  $8 \times 8$ )'

$$\begin{pmatrix} -.6 & .6 & -.6 & .6 & -.6 & .6 & .6 & -.6 \\ -.14 & -1.94 & .6 & 1.48 & -.14 & -1.94 & 1.48 & .6 \\ .88 & -.14 & -.6 & -.14 & -.14 & 1.48 & -1.94 & .6 \\ -.14 & 1.48 & .6 & -1.94 & .88 & -.14 & -.14 & -.6 \\ -1.94 & -.14 & .6 & 1.48 & -1.94 & -.14 & 1.48 & .6 \\ .6 & -.6 & -.6 & .6 & .6 & -.6 & .6 & -.6 \\ -.14 & .88 & -.6 & -.14 & 1.48 & -.14 & -1.94 & .6 \\ 1.48 & -.14 & .6 & -1.94 & -.14 & .88 & -.14 & -.6 \end{pmatrix} \quad (5)$$

Lower right  $8 \times 8$

$$\begin{pmatrix} 4.48 & -1.94 & -.6 & -1.94 & -1.94 & 1.48 & -.14 & .6 \\ & 4.48 & -.6 & -1.94 & 1.48 & -1.94 & -.14 & .6 \\ & & 1.8 & -.6 & .6 & .6 & -.6 & -.6 \\ & & & 4.48 & -.14 & -.14 & .88 & -.6 \\ & & & & 4.48 & -1.94 & -1.94 & -.6 \\ & & & & & 4.48 & -1.94 & -.6 \\ & & & & & & 4.48 & -.6 \\ & & & & & & & 1.8 \end{pmatrix} \quad (6)$$

A solution to these equations is

$$\begin{aligned}\mu^o &= 0, \\ \mathbf{s}^o &= (2.923, 1.713, 2.311, 2.329)', \\ \mathbf{d}^o &= (-.652, -.636, -.423, 0)', \\ p^o &= .007.\end{aligned}$$

$\hat{\gamma}$  in tabular form is

$$\begin{pmatrix} -1.035 & -.754 & -1.749 & 3.538 \\ .898 & .377 & -.434 & -.841 \\ 1.286 & -.836 & 1.453 & -1.902 \\ -1.149 & 1.214 & .729 & -.795 \end{pmatrix}.$$

The resulting  $\hat{\mu}_{ij}$  are

$$\begin{pmatrix} 1.243 & 1.533 & .752 & 6.462 \\ 1.959 & 1.461 & .857 & .872 \\ 2.945 & .839 & 3.349 & .409 \\ .528 & 2.908 & 2.635 & 1.541 \end{pmatrix}.$$

Note that these  $\hat{\mu}_{ij} \neq \bar{y}_{ij}$  but are not markedly different from them. The same  $\hat{\mu}_{ij}$  can be obtained by using

$$\begin{aligned}\text{Var}(\boldsymbol{\gamma}_{ii}) &= -2.88 \mathbf{I}, \\ \text{Var}(\boldsymbol{\gamma}_{ij}) &= 7.2 \mathbf{I}, \\ \text{Var}(\mathbf{r}) &= 2.64 \mathbf{I}.\end{aligned}$$

The solution to these mixed model equations is different from before, but the resulting  $\hat{\mu}_{ij}$  are identical. Ordinarily one would not accept a negative “variance”. The reason for this in our example was a bad choice of the parameters of  $\mathbf{P}$ . The OLS coefficient matrix for this solution is in (30.7) ... (30.9). The right hand sides are (18, 22, 23, 22, 25, 16, 22, 22, 6, 3, 2, 7, 8, 3, 5, 6, 9, 4, 7, 3, 2, 6, 8, 6, 11, 11, 9, 9, 12, 11).  $\mu^o$  and  $d_4^o$  are deleted giving a solution of 0 for them. The OLS equations for the preceding method are the same as these except the last 6 equations and unknowns are deleted. The solution is

$$\begin{aligned}\mu^o &= 0, \\ \mathbf{s}^o &= (1.460, 1.379, 2.838, 1.058)', \\ \mathbf{d}^o &= (-.844, .301, 1.375, 0)', \\ p^o &= .007, \\ \mathbf{r}^o &= (.253, -.239, 1.125, -.888, .220, -.471)'. \end{aligned}$$

$$\hat{\gamma} \text{ in tabular form} = \begin{pmatrix} .621 & -.481 & -1.844 & 3.877 \\ 1.172 & -.226 & -1.009 & -.727 \\ 1.191 & -1.412 & -.872 & -1.958 \\ -.810 & 1.328 & .673 & .477 \end{pmatrix}.$$

This solution gives the same result for  $\hat{\mu}_{ij}$  as before.

Upper left  $15 \times 15$

$$\begin{pmatrix} 11 & 0 & 0 & 0 & 5 & 2 & 3 & 5 & 5 & 2 & 3 & 1 & 0 & 0 & 0 \\ & 19 & 0 & 0 & 4 & 2 & 6 & 2 & 0 & 0 & 0 & 0 & 4 & 2 & 6 \\ & & 18 & 0 & 3 & 5 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 13 & 4 & 2 & 3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & 16 & 0 & 0 & 5 & 5 & 0 & 0 & 0 & 4 & 0 & 0 \\ & & & & & 11 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 \\ & & & & & & 14 & 2 & 0 & 0 & 3 & 0 & 0 & 0 & 6 \\ & & & & & & & 13 & 5 & 0 & 0 & 0 & 0 & 2 & 0 \\ & & & & & & & & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & & & 2 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & & & & 3 & 0 & 0 & 0 & 0 \\ & & & & & & & & & & & 1 & 0 & 0 & 0 \\ & & & & & & & & & & & & 4 & 0 & 0 \\ & & & & & & & & & & & & & 2 & 0 \\ & & & & & & & & & & & & & & 6 \end{pmatrix}. \quad (7)$$

Upper right  $15 \times 15$  and (lower left  $15 \times 15$ )'

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 1 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 6 & 7 & 0 \\ 0 & 3 & 5 & 2 & 8 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 5 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 4 & 2 & 3 & 4 & 0 & 0 & 4 & 0 & 2 & 3 \\ 0 & 3 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 & 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 5 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 3 & 0 & 0 & 3 & 0 & 6 & 0 & 3 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \end{pmatrix}. \quad (8)$$

Lower right  $15 \times 15$

$$\begin{pmatrix} 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 \\ & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ & & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ & & & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \\ & & & & & 4 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ & & & & & & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ & & & & & & & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ & & & & & & & & \text{dg} & (4 & 6 & 6 & 5 & 11 & 9 & 11) \end{pmatrix} \quad (9)$$

The method just preceding is convenient for missing subclasses. In that case  $\gamma_{ij}$  associated with  $n_{ij} = 0$  are included in the mixed model equations.