Chapter 24
Animal Model, Repeated Records

C. R. Henderson

1984 - Guelph

In this chapter we deal with a one trait, repeated records model that has been extensively used in animal breeding, and particularly in lactation studies with dairy cattle. The assumptions of this model are not entirely realistic, but may be an adequate approximation. The scalar model is

$$y_{ij} \;=\; \mathbf{x}_{ij}^{'}\boldsymbol{\beta} \;+\; \mathbf{z}_{ij}^{'}\mathbf{u} \;+\; c_i \;+\; e_{ij}. \tag{1}$$

- $\boldsymbol{\beta}$ represents fixed effects, and $\mathbf{x}_{ij}^{'}$ relates the j^{th} record of the i^{th} animal to elements of $\boldsymbol{\beta}$.
- $\mathbf{u}$ represents other random effects, and $\mathbf{z}_{ij}^{'}$ relates the record to them.
- c_i is a "cow" effect. It represents both genetic merit for production and permanent environmental effects.
- e_{ij} is a random "error" associated with the individual record.

The vector representation is

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \mathbf{Z}_c\mathbf{c} + \mathbf{e}. \tag{2}$$

$$\begin{aligned}
Var(\mathbf{u}) &= \mathbf{G}, \\
Var(\mathbf{c}) &= \mathbf{I}\ \sigma_c^2 \text{ if cows are unrelated, with } \sigma_c^2 = \sigma_a^2 + \sigma_p^2 \\
&= \mathbf{A}\ \sigma_a^2 + \mathbf{I}\ \sigma_p^2 \text{ if cows are related,}
\end{aligned}$$

where σ_p^2 is the variance of permanent environmental effects, and if there are non-additive genetic effects, it also includes their variances. In that case $\mathbf{I}\ \sigma_p^2$ is only approximate.

$$Var(\mathbf{e}) = \mathbf{I}\ \sigma_e^2.$$

$Cov(\mathbf{u}, \mathbf{a}')$, $Cov(\mathbf{u}, \mathbf{e}')$, and $Cov(\mathbf{a}, \mathbf{e}')$ are all null. For the related cow model let

$$\mathbf{Z}_c\mathbf{c} = \mathbf{Z}_c\mathbf{a} + \mathbf{Z}_c\mathbf{p}. \tag{3}$$

It is advantageous to use this latter model in setting up the mixed model equations, for then the simple method for computing $\mathbf{A}^{-1}$ can be used. There appears to be no simple method for computing directly the inverse of $Var(\mathbf{c})$.

$$\begin{pmatrix} \mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{Z} & \mathbf{X}'\mathbf{Z}_c & \mathbf{X}'\mathbf{Z}_c \\ \mathbf{Z}'\mathbf{X} & \mathbf{Z}'\mathbf{Z}+\sigma_e^2\mathbf{G}^{-1} & \mathbf{Z}'\mathbf{Z}_c & \mathbf{Z}'\mathbf{Z}_c \\ \mathbf{Z}_c'\mathbf{X} & \mathbf{Z}_c'\mathbf{Z} & \mathbf{Z}_c'\mathbf{Z}_c+\mathbf{A}^{-1}\frac{\sigma_e^2}{\sigma_a^2} & \mathbf{Z}_c'\mathbf{Z}_c \\ \mathbf{Z}_c'\mathbf{X} & \mathbf{Z}_c'\mathbf{Z} & \mathbf{Z}_c'\mathbf{Z}_c & \mathbf{Z}_c'\mathbf{Z}_c+\mathbf{I}\frac{\sigma_e^2}{\sigma_p^2} \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}^o \\ \hat{\mathbf{u}} \\ \hat{\mathbf{a}} \\ \hat{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \mathbf{X}'\mathbf{y} \\ \mathbf{Z}'\mathbf{y} \\ \mathbf{Z}_c'\mathbf{y} \\ \mathbf{Z}_c'\mathbf{y} \end{pmatrix} \quad (4)$$

These equations are easy to write provided $\mathbf{G}^{-1}$ is easy to compute, $\mathbf{G}$ being diagonal, e.g. as is usually the case. $\mathbf{A}^{-1}$ can be computed by the easy method. Further $\mathbf{Z}_c'\mathbf{Z}_c+\mathbf{I}\sigma_e^2/\sigma_p^2$ is diagonal, so $\hat{\mathbf{p}}$ can be "absorbed" easily. In fact, one would not need to write the $\hat{\mathbf{p}}$ equations. See Henderson (1975b). Also $\mathbf{Z}'\mathbf{Z}+\sigma_e^2\mathbf{G}^{-1}$ is sometimes diagonal and therefore $\hat{\mathbf{u}}$ can be absorbed easily. If predictions of breeding values are of primary interest, $\hat{\mathbf{a}}$ is what is wanted. If, in addition, predictions of real producing abilities are wanted, one needs $\hat{\mathbf{p}}$. Note that by subtracting the 4^{th} equation of (24.4) from the 3^{rd} we obtain

$$\mathbf{A}^{-1}\left(\sigma_e^2/\sigma_a^2\right)\ \hat{\mathbf{a}} - \mathbf{I}\left(\sigma_e^2/\sigma_p^2\right)\ \hat{\mathbf{p}} \ = \ \mathbf{0}.$$

Consequently

$$\hat{\mathbf{p}} \ = \ \left(\sigma_p^2/\sigma_a^2\right)\ \mathbf{A}^{-1}\hat{\mathbf{a}}, \quad (5)$$

and predictions of real producing abilities are

$$\left(\mathbf{I}+\left(\sigma_p^2/\sigma_a^2\right)\ \mathbf{A}^{-1}\right)\ \hat{\mathbf{a}}. \quad (6)$$

Note that under the model used in this chapter

$$Var(y_{ij}) \ = \ Var(y_{ik}),\ j \neq k.$$

$Cov(y_{ij}, y_{ik})$ is identical for all pairs of $j \neq k$. This is not necessarily a realistic model. If we wish a more general model, probably the most logical and easiest one to analyze is that which treats different lactations as separate traits, the methods for which are described in Chapter 26.

We illustrate the simple repeatability model with the following example. Four animals produced records as follows in treatments 1,2,3. The model is

$$y_{ij} \ = \ t_i \ + \ a_j \ + \ p_j \ + \ e_{ij}.$$

	Animals			
Treatment	1	2	3	4
1	5	3	-	4
2	6	5	7	-
3	8	-	9	-

The relationship matrix of the 4 animals is

$$\begin{pmatrix} 1 & .5 & .5 & .5 \\ & 1 & .25 & .125 \\ & & 1 & .5 \\ & & & 1 \end{pmatrix}.$$

$$\begin{aligned} Var(\mathbf{a}) &= .25\ \mathbf{A}\sigma_y^2, \\ Var(\mathbf{p}) &= .2\ \mathbf{I}\sigma_y^2, \\ \mathbf{I}\sigma_e^2 &= .55\ \mathbf{I}\sigma_y^2. \end{aligned}$$

These values correspond to $h^2 = .25$ and $r = .45$, where r denotes repeatability. The OLS equations are

$$\begin{pmatrix} 3 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ & 3 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ & & 2 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ & & & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ & & & & 2 & 0 & 0 & 0 & 2 & 0 & 0 \\ & & & & & 2 & 0 & 0 & 0 & 2 & 0 \\ & & & & & & 1 & 0 & 0 & 0 & 1 \\ & & & & & & & 3 & 0 & 0 & 0 \\ & & & & & & & & 2 & 0 & 0 \\ & & & & & & & & & 2 & 0 \\ & & & & & & & & & & 1 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{a} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} 12 \\ 18 \\ 17 \\ 19 \\ 8 \\ 16 \\ 4 \\ 19 \\ 8 \\ 16 \\ 4 \end{pmatrix}. \quad (7)$$

Note that the last 4 equations are identical to equations 4-7. Thus **a** and **p** are confounded in a fixed model. Now we add 2.2 $\mathbf{A}^{-1}$ to the 4-7 diagonal block of coefficients and 2.75 $\mathbf{I}$ to the 8-11 diagonal block of coefficients. The resulting coefficient matrix is in (24.8). $2.2 = .55/.25$, and $2.75 = .55/.2$.

$$\begin{pmatrix} 3.0 & 0 & 0 & 1.0 & 1.0 & 0 & 1.0 & 1.0 & 1.0 & 0 & 1.0 \\ & 3.0 & 0 & 1.0 & 1.0 & 1.0 & 0 & 1.0 & 1.0 & 1.0 & 0 \\ & & 2.0 & 1.0 & 0 & 1.0 & 0 & 1.0 & 0 & 1.0 & 0 \\ & & & 7.2581 & -1.7032 & -.9935 & -1.4194 & 3.0 & 0 & 0 & 0 \\ & & & & 5.0280 & -.1892 & .5677 & 0 & 2.0 & 0 & 0 \\ & & & & & 5.3118 & -1.1355 & 0 & 0 & 2.0 & 0 \\ & & & & & & 4.4065 & 0 & 0 & 0 & 1.0 \\ & & & & & & & 5.75 & 0 & 0 & 0 \\ & & & & & & & & 4.75 & 0 & 0 \\ & & & & & & & & & 4.75 & 0 \\ & & & & & & & & & & 3.75 \end{pmatrix} \quad (8)$$

The inverse of (24.8) (times 1000) is

$$\begin{pmatrix} 693 & 325 & 313 & -280 & -231 & -217 & -247 & -85 & -117 & -43 & -119 \\ & 709 & 384 & -288 & -246 & -266 & -195 & -96 & -114 & -118 & -34 \\ & & 943 & -306 & -205 & -303 & -215 & -126 & -60 & -152 & -26 \\ & & & 414 & 227 & 236 & 225 & -64 & 24 & 26 & 15 \\ & & & & 390 & 153 & 107 & 0 & -64 & 31 & 33 \\ & & & & & 410 & 211 & 14 & 37 & -53 & 2 \\ & & & & & & 406 & -3 & 48 & -2 & -42 \\ & & & & & & & 261 & 38 & 41 & 24 \\ & & & & & & & & 286 & 21 & 18 \\ & & & & & & & & & 290 & 12 \\ & & & & & & & & & & 310 \end{pmatrix} \tag{9}$$

The solution is

$$\begin{aligned} \hat{\mathbf{t}}' &= (4.123 \;\; 5.952 \;\; 8.133), \\ \hat{\mathbf{a}}' &= (.065, \; -.263, \; .280, \; .113), \\ \hat{\mathbf{p}}' &= (.104, \; -.326, \; .285, \; -.063). \end{aligned}$$

We next estimate σ_e^2, σ_a^2, σ_p^2, by MIVQUE with the priors that were used in the above mixed model solution. The $\mathbf{Z}_c'\mathbf{W}$ submatrix for both $\mathbf{a}$ and $\mathbf{p}$ is

$$\begin{pmatrix} 1 & 1 & 1 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

The variance of the right hand sides of the mixed model equations contains $\mathbf{W}'\mathbf{Z}_c\mathbf{A}\mathbf{Z}_c'\mathbf{W}\ \sigma_a^2$, where $\mathbf{W} = (\mathbf{X}\ \mathbf{Z}\ \mathbf{Z}_c\ \mathbf{Z}_c)$. The matrix of coefficients of σ_a^2 is in (24.11). $Var(\mathbf{r})$ also contains $\mathbf{W}'\mathbf{Z}_c\mathbf{Z}_c'\mathbf{W}\ \sigma_p^2$ and this matrix is in (24.12). The coefficients of σ_e^2 are in (24.7).

$$\begin{pmatrix} 5.25 & 4.88 & 3.25 & 6.0 & 3.25 & 2.5 & 1.65 & 6.0 & 3.25 & 2.5 & 1.63 \\ & 5.5 & 3.75 & 6.0 & 3.5 & 3.5 & 1.13 & 6.0 & 3.5 & 3.5 & 1.13 \\ & & 3.0 & 4.5 & 1.5 & 3.0 & 1.0 & 4.5 & 1.5 & 3.0 & 1.0 \\ & & & 9.0 & 3.0 & 3.0 & 1.5 & 9.0 & 3.0 & 3.0 & 1.5 \\ & & & & 4.0 & 1.0 & .25 & 3.0 & 4.0 & 1.0 & .25 \\ & & & & & 4.0 & 1.0 & 3.0 & 1.0 & 4.0 & 1.0 \\ & & & & & & 1.0 & 1.5 & .25 & 1.0 & 1.0 \\ & & & & & & & 9.0 & 3.0 & 3.0 & 1.5 \\ & & & & & & & & 4.0 & 1.0 & .25 \\ & & & & & & & & & 4.0 & 1.0 \\ & & & & & & & & & & 1.0 \end{pmatrix} \tag{11}$$

$$\begin{pmatrix} 3 & 2 & 1 & 3 & 2 & 0 & 1 & 3 & 2 & 0 & 1 \\ & 3 & 2 & 3 & 2 & 2 & 0 & 3 & 2 & 2 & 0 \\ & & 2 & 3 & 0 & 2 & 0 & 3 & 0 & 2 & 0 \\ & & & 9 & 0 & 0 & 0 & 9 & 0 & 0 & 0 \\ & & & & 4 & 0 & 0 & 0 & 4 & 0 & 0 \\ & & & & & 4 & 0 & 0 & 0 & 4 & 0 \\ & & & & & & 1 & 0 & 0 & 0 & 1 \\ & & & & & & & 9 & 0 & 0 & 0 \\ & & & & & & & & 4 & 0 & 0 \\ & & & & & & & & & 4 & 0 \\ & & & & & & & & & & 1 \end{pmatrix} \tag{12}$$

Now $Var(\hat{\mathbf{a}})$ contains $\mathbf{C}_a(Var(\mathbf{r}))\mathbf{C}_a'\sigma_a^2$, where $\mathbf{C}_a$ is the matrix formed by rows 4-9 of the matrix in (24.9). Then $\mathbf{C}_a(Var(\mathbf{r}))\mathbf{C}_a^{'}$ is

$$\begin{pmatrix} .0168 & .0012 & -.0061 & .0012 \\ & .0423 & -.0266 & -.0323 \\ & & .0236 & .0160 \\ & & & .0274 \end{pmatrix} \sigma_a^2 \tag{13}$$

$$+ \begin{pmatrix} .0421 & -.0019 & -.0099 & .0050 \\ & .0460 & -.0298 & -.0342 \\ & & .0331 & .0136 \\ & & & .0310 \end{pmatrix} \sigma_p^2 \tag{14}$$

$$+ \begin{pmatrix} .0172 & .0001 & -.0022 & -.0004 \\ & .0289 & -.0161 & -.0234 \\ & & .0219 & .0042 \\ & & & .0252 \end{pmatrix} \sigma_e^2. \tag{15}$$

We need $\hat{\mathbf{a}}'\mathbf{A}^{-1}\hat{\mathbf{a}}' \ = \ .2067$. The expectation of this is

$$\begin{aligned} tr\mathbf{A}^{-1} & \ [\text{matrix } (24.13) \ + \ \text{matrix } (24.14) \ + \ \text{matrix } (24.15)] \\ & = .1336 \ \sigma_e^2 \ + \ .1423 \ \sigma_a^2 \ + \ .2216 \ \sigma_p^2. \end{aligned}$$

To find $Var(\hat{\mathbf{p}})$ we use $\mathbf{C}_p$, the last 6 rows of (24.9).

$$Var(\hat{\mathbf{p}}) \ = \ \begin{pmatrix} .0429 & -.0135 & -.0223 & -.0071 \\ & .0455 & -.0154 & -.0166 \\ & & .0337 & .0040 \\ & & & .0197 \end{pmatrix} \sigma_a^2 \tag{16}$$

$$+ \begin{pmatrix} .1078 & -.0423 & -.0466 & -.0189 \\ & .0625 & -.0106 & -.0096 \\ & & .0586 & -.0014 \\ & & & .0298 \end{pmatrix} \sigma_p^2 \tag{17}$$

$$+ \begin{pmatrix} .0441 & -.0167 & -.0139 & -.0135 \\ & .0342 & -.0101 & -.0074 \\ & & .0374 & -.0133 \\ & & & .0341 \end{pmatrix} \sigma_e^2. \qquad (18)$$

We need $\hat{\mathbf{p}}'\hat{\mathbf{p}} \ = \ .2024$ with expectation

$$\begin{aligned} & tr[\text{matrix } (24.16) \ + \ \text{matrix } (24.17) \ + \ \text{matrix } (24.18)] \\ & \quad = \ .1498 \ \sigma_e^2 \ + \ .1419 \ \sigma_a^2 \ + \ .2588 \ \sigma_p^2. \end{aligned}$$

We need $\hat{\mathbf{e}}'\hat{\mathbf{e}}$.

$$\hat{\mathbf{e}} \ = \ [\mathbf{I} - \mathbf{WCW}']\mathbf{y},$$

where $\mathbf{C}$ = matrix (24.9), and $\mathbf{I} - \mathbf{WCW}'$ is

$$\begin{pmatrix} .4911 & -.2690 & -.2221 & -.1217 & .1183 & .0034 & -.0626 & .0626 \\ & .4548 & -.1858 & .1113 & -.1649 & .0536 & .0289 & -.0289 \\ & & .4079 & .0104 & .0466 & -.0570 & .0337 & -.0337 \\ & & & .5122 & -.2548 & -.2574 & -.1152 & .1152 \\ & & & & .4620 & -.2073 & -.0238 & .0238 \\ & & & & & .4647 & .1390 & -.1390 \\ & & & & & & .3729 & -.3729 \\ & & & & & & & .3729 \end{pmatrix} \qquad (19)$$

Then

$$\begin{aligned} \hat{\mathbf{e}} &= [.7078, \ -.5341, \ -.1736, \ -.1205, \ -.3624, \ .4829, \ -.3017, \ .3017]. \\ \hat{\mathbf{e}}'\hat{\mathbf{e}} &= 1.3774. \\ Var(\hat{\mathbf{e}}) &= (\mathbf{I} - \mathbf{WCW}') \ Var(\mathbf{y}) \ (\mathbf{I} - \mathbf{WCW}'), \\ Var(\mathbf{y}) &= \mathbf{Z}_c\mathbf{A}\mathbf{Z}_c' \ \sigma_a^2 + \mathbf{Z}_c\mathbf{Z}_c' \ \sigma_p^2 \ + \mathbf{I} \ \sigma_e^2. \\ &= \begin{pmatrix} 1 & .5 & .5 & 1. & .5 & .5 & 1. & .5 \\ & 1. & .125 & .5 & 1. & .25 & .5 & .25 \\ & & 1. & .5 & .125 & .5 & .5 & .5 \\ & & & 1. & .5 & .5 & 1. & .5 \\ & & & & 1. & .25 & .5 & .25 \\ & & & & & 1. & .5 & 1. \\ & & & & & & 1. & .5 \\ & & & & & & & 1. \end{pmatrix} \sigma_a^2 \\ & \quad + \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 & 1 & 0 \\ & & & & 1 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 1 \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{pmatrix} \sigma_p^2 + \mathbf{I}\sigma_e^2. \end{aligned}$$

Then the diagonals of $Var(\hat{\mathbf{e}})$ are

$$\begin{aligned} & (.0651,\ .1047,\ .1491,\ .0493,\ .0918,\ .0916,\ .0475,\ .0475)\ \sigma_a^2 \\ + & (.1705,\ .1358,\ .2257,\ .1167,\ .1498,\ .1462,\ .0940,\ .0940)\ \sigma_p^2 \\ + & [\text{diagonals of } (24.19)]\ \sigma_e^2. \end{aligned}$$

Then $E(\hat{\mathbf{e}}'\hat{\mathbf{e}})$ is the sum of these diagonals

$$= \ .6465\ \sigma_a^2 \ + \ 1.1327\ \sigma_p^2 \ + \ 3.5385\ \sigma_e^2.$$