Chapter 23
Sire Model, Single Records

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A simple sire model is one in which sires, possibly related, are mated to a random sample of unrelated dams, no dam has more than one progeny with a record, and each progeny produces one record. A scalar model for this is

$$y_{ij} = \mathbf{x}_{ij}^{'}\boldsymbol{\beta} + \mathbf{s}_i + \mathbf{z}_i^{'}\mathbf{u} + \mathbf{e}_{ij}. \quad (1)$$

$\boldsymbol{\beta}$ represents fixed effects with $\mathbf{x}_{ij}$ relating the j^{th} progeny of the i^{th} sire to these effects.

$\mathbf{s}_i$ represents the sire effect on the progeny record.

$\mathbf{u}$ represents other random factors with $\mathbf{z}_{ij}$ relating these to the ij^{th} progeny record.

$\mathbf{e}_{ij}$ is a random "error".

The vector representation is

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}_s\mathbf{s} + \mathbf{Z}\mathbf{u} + \mathbf{e}. \quad (2)$$

$Var(\mathbf{s}) = \mathbf{A}\sigma_s^2$, where $\mathbf{A}$ is the numerator relationship of the sires, and σ_s^2 is the sire variance in the "base" population. If the sires comprise a random sample from this population $\sigma_s^2 = \frac{1}{4}$ additive genetic variance. Some columns of $\mathbf{Z}_s$ will be null if $\mathbf{s}$ contains sires with no progeny, as will usually be the case if the simple method for computation of $\mathbf{A}^{-1}$ requiring base population animals, is used.

$$\begin{aligned} Var(\mathbf{u}) &= \mathbf{G},\ Cov(\mathbf{s}, \mathbf{u}') = \mathbf{0}. \\ Var(\mathbf{e}) &= \mathbf{R}, \text{usually} = \mathbf{I}\sigma_e^2. \\ Cov(\mathbf{s}, \mathbf{e}') &= \mathbf{0}, Cov(\mathbf{u}, \mathbf{e}') = \mathbf{0}. \end{aligned}$$

If sires and dams are truly random,

$$\begin{aligned} \mathbf{I}\sigma_e^2 &= .75\mathbf{I} \text{ (additive genetic variance)} \\ &\quad + \mathbf{I} \text{ (environmental variance)}. \end{aligned}$$

With this model the mixed model equations are

$$\left(\begin{array}{lll} \mathbf{X}'\mathbf{R}^{-1}\mathbf{X} & \mathbf{X}'\mathbf{R}^{-1}\mathbf{Z}_s & \mathbf{X}'\mathbf{R}^{-1}\mathbf{Z} \\ \mathbf{Z}_s'\mathbf{R}^{-1}\mathbf{X} & \mathbf{Z}_s'\mathbf{R}^{-1}\mathbf{Z}_s + \mathbf{A}^{-1}\sigma_s^{-2} & \mathbf{Z}_s'\mathbf{R}^{-1}\mathbf{Z} \\ \mathbf{Z}'\mathbf{R}^{-1}\mathbf{X} & \mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z}_s & \mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z} + \mathbf{G}^{-1} \end{array} \right) \left(\begin{array}{c} \boldsymbol{\beta}^o \\ \hat{\mathbf{s}} \\ \hat{\mathbf{u}} \end{array} \right) = \left(\begin{array}{c} \mathbf{X}'\mathbf{R}^{-1}\mathbf{y} \\ \mathbf{Z}_s'\mathbf{R}^{-1}\mathbf{y} \\ \mathbf{Z}'\mathbf{R}^{-1}\mathbf{y} \end{array} \right). \quad (3)$$

If $\mathbf{R} = \mathbf{I}\sigma_e^2$, (23.3) simplifies to (23.4)

$$\left(\begin{array}{lll} \mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{Z}_s & \mathbf{X}'\mathbf{Z} \\ \mathbf{Z}_s'\mathbf{X} & \mathbf{Z}_s'\mathbf{Z}_s + \mathbf{A}^{-1}\sigma_e^2/\sigma_s^2 & \mathbf{Z}_s'\mathbf{Z} \\ \mathbf{Z}'\mathbf{X} & \mathbf{Z}'\mathbf{Z}_s & \mathbf{Z}'\mathbf{Z} + \sigma_e^2\mathbf{G}^{-1} \end{array} \right) \left(\begin{array}{c} \boldsymbol{\beta}^o \\ \hat{\mathbf{s}} \\ \hat{\mathbf{u}} \end{array} \right) = \left(\begin{array}{c} \mathbf{X}'\mathbf{y} \\ \mathbf{Z}_s'\mathbf{y} \\ \mathbf{Z}'\mathbf{y} \end{array} \right). \quad (4)$$

We illustrate this model with the following data.

	n_{ij}				$y_{ij.}$			
	Herds				Herds			
Sires	1	2	3	4	1	2	3	4
1	3	5	0	0	25	34	–	–
2	0	8	4	0	–	74	31	–
3	4	2	6	8	23	11	43	73

The model assumed is

$$\begin{aligned} y_{ijk} &= s_i + h_j + e_{ijk.} \\ Var(\mathbf{s}) &= \mathbf{A}\sigma_e^2/12, \\ Var(\mathbf{e}) &= \mathbf{I}\sigma_e^2. \end{aligned}$$

$$\mathbf{A} = \left(\begin{array}{ccc} 1.0 & .5 & .5 \\ & 1.0 & .25 \\ & & 1.0 \end{array} \right).$$

h is fixed.

The ordinary LS equations are

$$\left(\begin{array}{ccccccc} 8 & 0 & 0 & 3 & 5 & 0 & 0 \\ & 12 & 0 & 0 & 8 & 4 & 0 \\ & & 20 & 4 & 2 & 6 & 8 \\ & & & 7 & 0 & 0 & 0 \\ & & & & 15 & 0 & 0 \\ & & & & & 10 & 0 \\ & & & & & & 8 \end{array} \right) \left(\begin{array}{c} \hat{\mathbf{s}} \\ \hat{\mathbf{h}} \end{array} \right) = \left(\begin{array}{r} 59 \\ 105 \\ 150 \\ 48 \\ 119 \\ 74 \\ 73 \end{array} \right). \quad (5)$$

The mixed model equations are

$$\begin{pmatrix} 28 & -8 & -8 & 3 & 5 & 0 & 0 \\ & 28 & 0 & 0 & 8 & 4 & 0 \\ & & 36 & 4 & 2 & 6 & 8 \\ & & & 7 & 0 & 0 & 0 \\ & & & & 15 & 0 & 0 \\ & & & & & 10 & 0 \\ & & & & & & 8 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{s}} \\ \hat{\mathbf{h}} \end{pmatrix} = \begin{pmatrix} 59 \\ 105 \\ 150 \\ 48 \\ 119 \\ 74 \\ 73 \end{pmatrix}. \quad (6)$$

The inverse of the matrix of (23.6) is

$$\begin{pmatrix} .0764 & .0436 & .0432 & -.0574 & -.0545 & -.0434 & -.0432 \\ .0436 & .0712 & .0320 & -.0370 & -.0568 & -.0477 & -.0320 \\ .0432 & .0320 & .0714 & -.0593 & -.0410 & -.0556 & -.0714 \\ -.0574 & -.0370 & -.0593 & .2014 & .0468 & .0504 & .0593 \\ -.0545 & -.0568 & -.0410 & .0468 & .1206 & .0473 & .0410 \\ -.0434 & -.0477 & -.0556 & .0504 & .0473 & .1524 & .0556 \\ -.0432 & -.0320 & -.0714 & .0593 & .0410 & .0556 & .1964 \end{pmatrix}. \quad (7)$$

The solution is

$$\hat{\mathbf{s}}' = (-.036661, .453353, -.435022),$$
$$\hat{\mathbf{h}}' = (7.121439, 7.761769, 7.479672, 9.560022).$$

Let us estimate σ_e^2 from the residual mean square using OLS reduction, and σ_e^2 by MIVQUE type computations. A solution to the OLS equations is

$$[10.14097,\ 11.51238,\ 9.12500,\ -2.70328,\ -2.80359,\ -2.67995,\ 0]$$

This gives a reduction in SS of 2514.166.

$$\mathbf{y}'\mathbf{y} = 2922.$$

Then $\hat{\sigma}_e^2$ = (2922 - 2514.166)/(40-6) = 11.995. MIVQUE requires computation of $\hat{\mathbf{s}}'\mathbf{A}^{-1}\hat{\mathbf{s}}$ and equating to its expectation.

$$\mathbf{A}^{-1} = \frac{1}{3} \begin{pmatrix} 5 & -2 & -2 \\ -2 & 4 & 0 \\ -2 & 0 & 4 \end{pmatrix}.$$
$$\hat{\mathbf{s}}'\mathbf{A}^{-1}\hat{\mathbf{s}} = .529500.$$

Var (RHS of mixed model equations) = [Matrix (23.5)] σ_e^2 +

$$\begin{pmatrix} 8 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 20 \\ 3 & 0 & 4 \\ 5 & 8 & 2 \\ 0 & 4 & 6 \\ 0 & 0 & 8 \end{pmatrix} \mathbf{A} \begin{pmatrix} 8 & 0 & 0 & 3 & 5 & 0 & 0 \\ 0 & 12 & 0 & 0 & 8 & 4 & 0 \\ 0 & 0 & 20 & 4 & 2 & 6 & 8 \end{pmatrix} \sigma_s^2.$$

The second term of this is

$$\begin{pmatrix} 64 & 48 & 80 & 40 & 80 & 40 & 32 \\ & 144 & 60 & 30 & 132 & 66 & 24 \\ & & 400 & 110 & 130 & 140 & 160 \\ & & & 37 & 56 & 43 & 44 \\ & & & & 151 & 83 & 5 \\ & & & & & 64 & 56 \\ & & & & & & 64 \end{pmatrix} \sigma_s^2. \qquad (8)$$

$$\begin{aligned} Var(\hat{\mathbf{s}}) &= \mathbf{C}_s \text{ [matrix (23.8)] } \mathbf{C}_s' \, \sigma_s^2 \; + \; \mathbf{C}_s \text{ [matrix (23.5)] } \mathbf{C}_s' \, \sigma_e^2, \\ \text{where } \mathbf{C}_s &= \text{first 3 rows of (23.7).} \end{aligned}$$

$$\begin{aligned} Var(\hat{\mathbf{s}}) \;=\; & \begin{pmatrix} .005492 & -.001451 & -.001295 \\ & .007677 & -.006952 \\ & & .007599 \end{pmatrix} \sigma_e^2 \\ & + \begin{pmatrix} .017338 & -.005622 & -.003047 \\ & .053481 & -.050670 \\ & & .052193 \end{pmatrix} \sigma_s^2. \end{aligned} \qquad (9)$$

Then $E(\hat{\mathbf{s}}'\mathbf{A}^{-1}\hat{\mathbf{s}}) = tr(\mathbf{A}^{-1}$ [matrix (23.9)]$) = .033184 \, \sigma_e^2 + .181355 \, \sigma_s^2$. With these results we solve for $\hat{\sigma}_s^2$ and this is .7249 using estimated $\hat{\sigma}_e^2$ as 11.995. This is an approximate MIVQUE solution because $\hat{\sigma}_e^2$ was computed from the residual of ordinary least squares reduction rather than by MIVQUE.