Chapter 21
Analysis of Covariance Model

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1984 - Guelph

A covariance model is one in which **X** has columns referring to levels of factors or interactions and one or more columns of covariates. The model may or may not contain **Zu**. It usually does not in text book discussions of covariance models, but in animal breeding applications there would be, or at least should be, a **u** vector, usually referring to breeding values.

1 Two Way Fixed Model With Two Covariates

Consider a model

$$y_{ijk} = r_i + c_j + \gamma_{ij} + w_{1ijk}\alpha_1 + w_{2ijk}\alpha_2 + e_{ijk}.$$

All elements of the model are fixed except for **e**, which is assumed to have variance, $\mathbf{I}\sigma_e^2$. The n_{ijk}, $y_{ijk.}$, $w_{1ijk.}$, and $w_{2ijk.}$ are as follows

	n_{ijk}			$y_{ijk.}$			$w_{1ijk.}$			$w_{2ijk.}$		
	1	2	3	1	2	3	1	2	3	1	2	3
1	3	2	1	20	9	4	8	7	2	12	11	4
2	1	3	4	3	20	24	6	10	11	5	15	14
3	2	1	2	13	7	8	7	2	4	9	2	7

and the necessary sums of squares and crossproducts are

$$\begin{aligned}
\textstyle\sum_i \sum_j \sum_k w_{1ijk}^2 &= 209, \\
\textstyle\sum_i \sum_j \sum_k w_{1ijk}w_{2ijk} &= 264, \\
\textstyle\sum_i \sum_j \sum_k w_{2ijk}^2 &= 373, \\
\textstyle\sum_i \sum_j \sum_k w_{1ijk}y_{ijk} &= 321, \\
\textstyle\sum_i \sum_j \sum_k w_{2ijk}y_{ijk} &= 433.
\end{aligned}$$

Then the matrix of coefficients of OLS equations are in (21.1). The right hand side vector is (33, 47, 28, 36, 36, 36, 20, 9, 4, 3, 20, 24, 13, 7, 8, 321, 433)'.

$$
\begin{pmatrix}
6 & 0 & 0 & 3 & 2 & 1 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 17 & 27 \\
 & 8 & 0 & 1 & 3 & 4 & 0 & 0 & 0 & 1 & 3 & 4 & 0 & 0 & 0 & 27 & 34 \\
 & & 5 & 2 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 2 & 13 & 18 \\
 & & & 6 & 0 & 0 & 3 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 21 & 26 \\
 & & & & 6 & 0 & 0 & 2 & 0 & 0 & 3 & 0 & 0 & 1 & 0 & 19 & 28 \\
 & & & & & 7 & 0 & 0 & 1 & 0 & 0 & 4 & 0 & 0 & 2 & 17 & 25 \\
 & & & & & & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 12 \\
 & & & & & & & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 11 \\
 & & & & & & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 \\
 & & & & & & & & & 1 & 0 & 0 & 0 & 0 & 0 & 6 & 5 \\
 & & & & & & & & & & 3 & 0 & 0 & 0 & 0 & 10 & 15 \\
 & & & & & & & & & & & 4 & 0 & 0 & 0 & 11 & 14 \\
 & & & & & & & & & & & & 2 & 0 & 0 & 7 & 9 \\
 & & & & & & & & & & & & & 1 & 0 & 2 & 2 \\
 & & & & & & & & & & & & & & 2 & 4 & 7 \\
 & & & & & & & & & & & & & & & 209 & 264 \\
 & & & & & & & & & & & & & & & & 373
\end{pmatrix} \quad (1)
$$

A g-inverse of the coefficient matrix can be obtained by taking a regular inverse with the first 6 rows and columns set to 0. The lower 11 × 11 submatrix of the g-inverse is in (21.2).

$$
10^{-4}
\begin{pmatrix}
8685 & 7311 & 5162 & 7449 & 6690 & 4802 & 6163 & 2866 & 4588 & -285 & -1148 \\
 & 15009 & 7135 & 9843 & 9139 & 6507 & 8357 & 3832 & 6309 & -264 & -1652 \\
 & & 15313 & 5849 & 6452 & 4422 & 5694 & 2430 & 4592 & 226 & -1441 \\
 & & & 25714 & 9312 & 7519 & 9581 & 5325 & 5718 & 2401 & -262 \\
 & & & & 11695 & 6002 & 7704 & 3582 & 5735 & -356 & -1435 \\
 & & & & & 6938 & 5686 & 2780 & 4012 & -569 & -821 \\
 & & & & & & 12286 & 3551 & 5158 & -704 & -1072 \\
 & & & & & & & 11869 & 2290 & -654 & -281 \\
 & & & & & & & & 9016 & 6 & -1151 \\
 & & & & & & & & & 767 & -440 \\
 & & & & & & & & & & 580
\end{pmatrix} \quad (2)
$$

This gives a solution vector (0, 0, 0, 0, 0, 0, 7.9873, 6.4294, 5.7748, 2.8341, 8.3174, 6.8717, 7.6451, 7.2061, 5.3826, .6813, -.7843). One can test an hypothesis concerning interactions by subtracting from the reduction under the full model the reduction when $\boldsymbol{\gamma}$ is dropped from the model. This tests that all $\gamma_{ij} - \bar{\gamma}_{i.} - \bar{\gamma}_{.j} + \bar{\gamma}_{..}$ are 0. The reduction under the full model is 652.441. A solution with $\boldsymbol{\gamma}$ dropped is

$$(6.5808,\ 7.2026,\ 6.5141,\ 1.4134,\ 1.4386,\ 0,\ .1393,\ -.5915).$$

This gives a reduction = 629.353. Then the numerator SS with 4 d.f. is 652.441 - 629.353.

The usual test of hypothesis concerning rows is that all $r_i + \bar{c}_. + \bar{\gamma}_{i.}$ are equal. This is comparable to the test effected by weighted squares of means when there are no covariates. We could define the test as all $r_i + \bar{c}_. + \bar{\gamma}_{i.} + \alpha_1 w_{10} + \alpha_2 w_{20}$ are equal, where w_{10}, w_{20} can have any values. This is not valid, as shown in Section 16.6, when the regressions are not homogeneous. To find the numerator SS with 2 d.f. for rows take the matrix

$$\begin{aligned}
\mathbf{K}' &= \begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \end{pmatrix}. \\
\mathbf{K}'\hat{\boldsymbol{\gamma}} &= \begin{pmatrix} 2.1683 \\ -.0424 \end{pmatrix},
\end{aligned}$$

where $\hat{\boldsymbol{\gamma}}$ is the solution under the full model with $\mathbf{r}^o$, $\mathbf{c}^o$ set to $\mathbf{0}$. Next compute $\mathbf{K}'$ [first 9 rows and columns of (21.2)] $\mathbf{K}$ as

$$= \begin{pmatrix} 4.5929 & 2.2362 \\ 2.2362 & 4.3730 \end{pmatrix}.$$

Then

$$\begin{aligned}
\text{numerator SS} &= (2.1683 \quad -.0424) \begin{pmatrix} 4.5929 & 2.2362 \\ 2.2362 & 4.3730 \end{pmatrix}^{-1} \begin{pmatrix} 2.1683 \\ -.0424 \end{pmatrix} \\
&= 1.3908.
\end{aligned}$$

If we wish to test w_1, compute as the numerator SS, with 1 d.f., .6813 $(.0767)^{-1}$ .6813, where

$$\hat{\alpha}_1 = .6813,\ Var(\hat{\alpha}_1) = .0767\ \sigma_e^2.$$

2 Two Way Fixed Model With Missing Subclasses

We found in Section 17.3 that the two way fixed model with interaction and with one or more missing subclasses precludes obtaining the usual estimates and tests of main effects and interactions. This is true also, of course, in the covariance model with missing subclasses for fixed by fixed classifications. We illustrate with the same example as before except that the (3,3) subclass is missing. The OLS equations are in (21.3). The right hand side vector is (33, 47, 20, 36, 36, 28, 20, 9, 4, 3, 20, 24, 13, 7, 0, 307, 406)′. Note that the equation for γ_{33} is included even though the subclass is missing.

$$\begin{pmatrix}
6 & 0 & 0 & 3 & 2 & 1 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 17 & 27 \\
 & 8 & 0 & 1 & 3 & 4 & 0 & 0 & 0 & 1 & 3 & 4 & 0 & 0 & 0 & 27 & 34 \\
 & & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 9 & 11 \\
 & & & 6 & 0 & 0 & 3 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 21 & 26 \\
 & & & & 6 & 0 & 0 & 2 & 0 & 0 & 3 & 0 & 0 & 1 & 0 & 19 & 28 \\
 & & & & & 5 & 0 & 0 & 1 & 0 & 0 & 4 & 0 & 0 & 0 & 13 & 18 \\
 & & & & & & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 12 \\
 & & & & & & & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 11 \\
 & & & & & & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 \\
 & & & & & & & & & 1 & 0 & 0 & 0 & 0 & 0 & 6 & 5 \\
 & & & & & & & & & & 3 & 0 & 0 & 0 & 0 & 10 & 15 \\
 & & & & & & & & & & & 4 & 0 & 0 & 0 & 11 & 14 \\
 & & & & & & & & & & & & 2 & 0 & 0 & 7 & 9 \\
 & & & & & & & & & & & & & 1 & 0 & 2 & 2 \\
 & & & & & & & & & & & & & & 0 & 0 & 0 \\
 & & & & & & & & & & & & & & & 199 & 249 \\
 & & & & & & & & & & & & & & & & 348
\end{pmatrix} \quad (3)$$

We use these equations to estimate a pseudo-variance, σ^2_γ to use in biased estimation with priors on $\boldsymbol{\gamma}$. We use Method 3. Reductions and expectations are

$$\begin{aligned}
\mathbf{y}'\mathbf{y} &= 638, \; E(\mathbf{y}'\mathbf{y}) = 17\,\sigma^2_e + 17\,\sigma^2_\gamma + q. \\
\text{Red (full)} &= 622.111, \; E() = 10\,\sigma^2_e + 17\,\sigma^2_\gamma + q. \\
\text{Red } (\mathbf{r}, \mathbf{c}, \gamma) &= 599.534, \; E() = 7\,\sigma^2_e + 12.6121\,\sigma^2_\gamma + q. \\
\text{q} &= \text{a quadratic in } \mathbf{r}, \mathbf{c}, \boldsymbol{\alpha}.
\end{aligned}$$

Solving we get $\hat{\sigma}^2_e = 2.26985$, $\hat{\sigma}^2_\gamma = 3.59328$ or a ratio of .632. Then we add .632 to each of the diagonal coefficients corresponding to γ equations in (21.3). A resulting solution is

$$\begin{gathered}
(6.6338,\ 6.1454,\ 7.3150,\ -.3217,\ .6457,\ 0,\ 1.3247,\ -.7287,\ -.5960, \\
-1.7830, 1.1870, .5960, .4583, -.4583, 0, .6179, -.7242)
\end{gathered}$$

The resulting biased estimates of $r_i + c_j + \gamma_{ij}$ given $w_1 = w_2 = 0$ are

$$\begin{pmatrix}
7.6368 & 6.5509 & 6.0378 \\
4.0407 & 7.9781 & 6.7414 \\
7.4516 & 7.5024 & 7.3150
\end{pmatrix} \quad (4)$$

The matrix of estimated mean squared errors obtained by pre and post multiplying a g-inverse of the coefficient matrix by

$$\mathbf{L}' = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & \vdots & & & & & & & & & & & & \vdots & & \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

is in (21.5).

$$10,000^{-1} \begin{pmatrix} 8778 & 7287 & 5196 & 8100 & 6720 & 4895 & 6151 & 3290 & 2045 \\ & 13449 & 6769 & 8908 & 8661 & 6017 & 7210 & 4733 & 2788 \\ & & 13215 & 4831 & 5606 & 4509 & 4931 & 2927 & 7191 \\ & & & 22170 & 9676 & 7524 & 9408 & 4825 & 1080 \\ & & & & 11201 & 6007 & 7090 & 4505 & 2540 \\ & & & & & 6846 & 5423 & 3214 & 3885 \\ & & & & & & 11244 & 4681 & 5675 \\ & & & & & & & 10880 & 6514 \\ & & & & & & & & 45120 \end{pmatrix} \quad (5)$$

To test that all $r_i + \bar{c}_. + \bar{\gamma}_{i.}$ are equal, use the matrix

$$\begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \end{pmatrix} \text{ with (21.4) and (21.5).}$$

Then the numerator SS is

$$(1.4652 \quad -2.0434) \begin{pmatrix} 4.4081 & 1.6404 \\ & 9.2400 \end{pmatrix}^{-1} \begin{pmatrix} 1.4652 \\ -2.0434 \end{pmatrix} = 1.2636.$$

The test is approximate because the MSE depends upon $\hat{\sigma}^2_\gamma/\hat{\sigma}^2_e = \sigma^2_\gamma/\sigma^2_e$. Further, the numerator is not distributed as χ^2.

3 Covariates All Equal At The Same Level Of A Factor

In some applications every $w_{ij} = w_i$ in a one-way covariate model,

$$y_{ij} = \mu + t_i + w_{ij}\gamma + e_{ij}$$

with all $w_{ij} = w_i$. For example, t_i might represent an animal in which there are several observations, y_{ij}, but the covariate is measured only once. This idea can be extended to multiple classifications. When the factor associated with the constant covariate is fixed, estimability problems exist, Henderson and Henderson (1979). In the one way case $t_i - t_j$ is not estimable and neither is γ.

We illustrate with a one-way case in which n_i = (3,2,4), w_i = (2,4,5), $\bar{y}_{i.}$ = (6,5,10). The OLS equations are

$$\begin{pmatrix} 9 & 3 & 2 & 4 & 34 \\ 3 & 3 & 0 & 0 & 6 \\ 2 & 0 & 2 & 0 & 8 \\ 4 & 0 & 0 & 4 & 20 \\ 34 & 6 & 8 & 20 & 144 \end{pmatrix} \begin{pmatrix} \mu \\ t_1 \\ t_2 \\ t_3 \\ \gamma \end{pmatrix} = \begin{pmatrix} 68 \\ 18 \\ 10 \\ 40 \\ 276 \end{pmatrix}. \quad (6)$$

Note that equations 2,3,4 sum to equation 1 and also (2 4 5) times these equations gives the last equation. Accordingly the coefficient matrix has rank only 3, the same as if there were no covariate. A solution is (0,6,5,10,0).

If t is random, there is no problem of estimability for then we need only to look at the rank of

$$\begin{pmatrix} 9 & 34 \\ 34 & 144 \end{pmatrix},$$

and that is 2. Consequently μ and γ are both estimable, and of course **t** is predictable. Let us estimate σ_t^2 and σ_e^2 by Method 3 under the assumption $Var(\mathbf{t}) = \mathbf{I}\sigma_t^2$, $Var(\mathbf{e}) = \mathbf{I}\sigma_e^2$. For this we need **y'y**, reduction under the full model, and Red (μ, γ).

$$\begin{aligned} \mathbf{y}'\mathbf{y} &= 601,\ E(\mathbf{y}'\mathbf{y}) = 9\ \sigma_e^2 + 9\ \sigma_t^2 + q. \\ \text{Red (full)} &= 558,\ E() = 3\ \sigma_e^2 + 9\ \sigma_t^2 + q. \\ \text{Red } (\mu, \gamma) &= 537.257,\ E() = 2\ \sigma_e^2 + 6.6\ \sigma_t^2 + q. \end{aligned}$$

q is a quadratic in μ, γ. This gives estimates $\hat{\sigma}_e^2 = 7.167$, $\hat{\sigma}_t^2 = 5.657$ or a ratio of 1.27.

Let us use 1 as a prior value of σ_e^2/σ_t^2 and estimate σ_t^2 by MIVQUE given that σ_e^2 = 7.167. We solve for $\hat{t}$ having added 1 to the diagonal coefficients of equations 2,3,4 of (21.6). This gives an inverse,

$$\begin{pmatrix} 4.24370 & -1.63866 & -.08403 & .72269 & -1.02941 \\ & .94958 & .15126 & -.10084 & .35294 \\ & & .54622 & .30252 & -.05882 \\ & & & .79832 & -.29412 \\ & & & & .27941 \end{pmatrix}. \quad (7)$$

The solution is (3.02521, .55462, -1.66387, 1.10924, 1.11765). From this $\hat{\mathbf{t}}'\hat{\mathbf{t}} = 4.30648$. To find its expectation we compute

$$tr(\mathbf{C}_t\ [\text{matrix (21.6)}]\ \mathbf{C}_t') = tr \begin{pmatrix} .01483 & -.04449 & .02966 \\ & .13347 & -.08898 \\ & & .05932 \end{pmatrix} = .20761,$$

which is the coefficient of σ_e^2 in $E(\hat{\mathbf{t}}'\hat{\mathbf{t}})$. $\mathbf{C}_t$ is the submatrix composed of rows 2-4 of (21.7).

$$tr(\mathbf{C}_t\mathbf{W}'\mathbf{Z}_t\mathbf{Z}_t'\mathbf{W}\mathbf{C}_t') = tr\begin{pmatrix} .03559 & -.10677 & .07118 \\ & .32032 & -.21354 \\ & & .14236 \end{pmatrix} = .49827,$$

the coefficient of σ_t^2 in $E(\hat{\mathbf{t}}'\hat{\mathbf{t}})$. $\mathbf{W}'\mathbf{Z}_t$ is the submatrix composed of cols. 2-4 of (21.6). This gives $\hat{\sigma}_t^2 = 5.657$ or $\hat{\sigma}_e^2/\hat{\sigma}_t^2 = 1.27$. If we do another MIVQUE estimation of σ_t^2, given $\sigma_e^2 = 7.167$ using the ratio, 1.27, the same estimate of σ_t^2 is obtained. Accordingly we have REML of σ_t^2, given σ_e^2. Notice also that this is the Method 3 estimate.

If $\mathbf{t}$ were actually fixed, but we use a pseudo-variance in the mixed model equations we obtain biased estimators. Using $\hat{\sigma}_e^2/\hat{\sigma}_t^2 = 1.27$,

$$\begin{aligned} \hat{\mu} + \hat{t}_i &= (3.53,\ 1.48,\ 4.04). \\ \hat{\mu} + \hat{t}_i + w_i\hat{\gamma}_{.} &= (5.78,\ 5.98,\ 9.67). \end{aligned}$$

Contrast this last with the corresponding OLS estimates of (6,5,10).

4 Random Regressions

It is reasonable to assume that regression coefficients are random in some models. For example, suppose we have a model,

$$y_{ij} = \mu + c_i + w_{ij}\gamma_i + e_{ij},$$

where y_{ij} is a yield observation on the j^{th} day for the i^{th} cow, w_{ij} is the day, and γ_i is a regression coefficient, linear slope of yield on time. Linearity is a reasonable assumption for a relatively short period following peak production. Further, it is obvious that γ_i is different from cow to cow, and if cows are random, γ_i is also random. Consequently we should make use of this assumption. The following example illustrates the method. We have 4 random cows with 3,5,6,4 observations respectively. The OLS equations are in (21.8).

$$\begin{pmatrix} 18 & 3 & 5 & 6 & 4 & 10 & 30 & 19 & 26 \\ & 3 & 0 & 0 & 0 & 10 & 0 & 0 & 0 \\ & & 5 & 0 & 0 & 0 & 30 & 0 & 0 \\ & & & 6 & 0 & 0 & 0 & 19 & 0 \\ & & & & 4 & 0 & 0 & 0 & 26 \\ & & & & & 38 & 0 & 0 & 0 \\ & & & & & & 190 & 0 & 0 \\ & & & & & & & 67 & 0 \\ & & & & & & & & 182 \end{pmatrix} \begin{pmatrix} \mu^o \\ \mathbf{c}^o \\ \boldsymbol{\gamma}^o \end{pmatrix} = \begin{pmatrix} 90 \\ 14 \\ 18 \\ 26 \\ 32 \\ 51 \\ 117 \\ 90 \\ 216 \end{pmatrix} \quad (8)$$

$$\begin{aligned} 10 &= w_{1.}, \\ 30 &= w_{2.},\ \text{etc.} \\ 38 &= \textstyle\sum_j w_{ij}^2,\ \text{etc.} \\ 51 &= \textstyle\sum_j w_{ij} y_{ij},\ \text{etc.} \end{aligned}$$

First let us estimate σ_e^2, σ_c^2, σ_γ^2 by Method 3. The necessary reductions and their expectations are

$$E \begin{pmatrix} \mathbf{y}'\mathbf{y} \\ \text{Red (full)} \\ \text{Red } (\mu, \mathbf{t}) \\ \text{Red } (\mu, \boldsymbol{\gamma}) \end{pmatrix} = \begin{pmatrix} 18 & 18 & 477 \\ 8 & 18 & 477 \\ 4 & 18 & 442.5 \\ 5 & 16.9031 & 477 \end{pmatrix} \begin{pmatrix} \sigma_e^2 \\ \sigma_c^2 \\ \sigma_\gamma^2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} 18\ \mu^2.$$

The reductions are (538, 524.4485, 498.8, 519.6894). This gives estimates $\hat{\sigma}_e^2 = 1.3552$, $\hat{\sigma}_c^2 = .6324$, $\hat{\sigma}_\gamma^2 = .5863$. Using the resulting ratios, $\hat{\sigma}_e^2/\hat{\sigma}_c^2 = 2.143$ and $\hat{\sigma}_e^2/\hat{\sigma}_\gamma^2 = 2.311$, the mixed model solution is

$$(2.02339,\ .11180,\ -.36513,\ -.09307,$$
$$.34639,\ .73548,\ .34970,\ .76934,\ .83764).$$

Covariance models are discussed also in Chapter 16.