Chapter 2
Linear Unbiased Estimation

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We are interested in linear unbiased estimators of $\boldsymbol{\beta}$ or of linear functions of $\boldsymbol{\beta}$, say $\mathbf{k}'\boldsymbol{\beta}$. That is, the estimator has the form, $\mathbf{a}'\mathbf{y}$, and E $(\mathbf{a}'\mathbf{y}) = \mathbf{k}'\boldsymbol{\beta}$, if possible. It is not necessarily the case that $\mathbf{k}'\boldsymbol{\beta}$ can be estimated unbiasedly. If $\mathbf{k}'\boldsymbol{\beta}$ can be estimated unbiasedly, it is called estimable. How do we determine estimability?

1 Verifying Estimability

$$E(\mathbf{a}'\mathbf{y}) = \mathbf{a}'\mathbf{X}\boldsymbol{\beta}.$$

Does this equal $\mathbf{k}'\boldsymbol{\beta}$? It will for any value of $\boldsymbol{\beta}$ if and only if $\mathbf{a}'\mathbf{X} = \mathbf{k}'$.

Consequently, if we can find any $\mathbf{a}$ such that $\mathbf{a}'\mathbf{X} = \mathbf{k}'$, then $\mathbf{k}'\boldsymbol{\beta}$ is estimable. Let us illustrate with

$$\mathbf{X} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \\ 1 & 3 & 6 \end{pmatrix}.$$

- Is β_1 estimable, that is, $(1 \;\; 0 \;\; 0)\ \boldsymbol{\beta}$ estimable? Let $\mathbf{a}' = (2 \quad -1 \;\; 0 \;\; 0)$ then

  $$\mathbf{a}'\mathbf{X} = (1 \;\; 0 \;\; 0) = \mathbf{k}'.$$

  Therefore, $\mathbf{k}'\boldsymbol{\beta}$ is estimable.

- Is $(0 \;\; 1 \;\; 2)\ \boldsymbol{\beta}$ estimable? Let $\mathbf{a}' = (-1 \;\; 1 \;\; 0 \;\; 0)$ then

  $$\mathbf{a}'\mathbf{X} = (0 \;\; 1 \;\; 2) = \mathbf{k}'.$$

  Therefore, it is estimable.

- Is β_2 estimable? No, because no $\mathbf{a}'$ exists such that $\mathbf{a}'\mathbf{X} = (0 \;\; 1 \;\; 0)$.

Generally it is easier to prove by the above method that an estimable function is indeed estimable than to prove that a non-estimable function is non-estimable. Accordingly, we consider other methods for determining estimability.

1.1 Second Method

Partition $\mathbf{X}$ as follows with possible re-ordering of columns.

$$\mathbf{X} = (\mathbf{X}_1 \quad \mathbf{X}_1\mathbf{L}),$$

where $\mathbf{X}_1$ has r linearly independent columns. Remember that $\mathbf{X}$ is $n \times p$ with rank $= r$. The dimensions of $\mathbf{L}$ are $r \times (p - r)$.

Then $\mathbf{k}'\boldsymbol{\beta}$ is estimable if and only if

$$\mathbf{k}' = (\mathbf{k}_1' \quad \mathbf{k}_1'\mathbf{L}),$$

where $\mathbf{k}_1'$ has r elements, and $\mathbf{k}_1'\mathbf{L}$ has $p - r$ elements. Consider the previous example.

$$\mathbf{X}_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \\ 1 & 3 \end{pmatrix}, \text{ and } \mathbf{L} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

- Is $(1 \quad 0 \quad 0)\ \boldsymbol{\beta}$ estimable?

  $$\mathbf{k}_1' = (1\ \ 0),\ \mathbf{k}_1'\mathbf{L} = (1\ \ 0) \quad \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad = 0.$$

  Thus $\mathbf{k}' = (1\ \ 0\ \ 0)$, and the function is estimable.

- Is $(0\ \ 1\ \ 2)\ \boldsymbol{\beta}$ estimable?

  $$\mathbf{k}_1' = (0\ \ 1), \text{ and } \mathbf{k}_1'\mathbf{L} = (0\ \ 1)\begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad = 2.$$

  Thus $\mathbf{k}' = (0\ \ 1\ \ 2)$, and the function is estimable.

- Is $(0\ \ 1\ \ 0)\ \boldsymbol{\beta}$ estimable?

  $$\mathbf{k}_1' = (0\ \ 1), \text{ and } \mathbf{k}_1'\mathbf{L} = (0\ \ 1)\begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad = 2.$$

  Thus $(\mathbf{k}_1' \quad \mathbf{k}_1'\mathbf{L}). = (0\ \ 1\ \ 2) \neq (0\ \ 1\ \ 0)$. The function is not estimable.

1.2 Third Method

A third method is to find a matrix, $\mathbf{C}$, of order $p \times (p - r)$ and rank, $p - r$, such that

$$\mathbf{XC} = \mathbf{0}.$$

Then $\mathbf{k}'\beta$ is estimable if and only if

$$\mathbf{k}'\mathbf{C} = \mathbf{0}.$$

In the example

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \\ 1 & 3 & 6 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

- Therefore $(1 \;\; 0 \;\; 0) \;\; \boldsymbol{\beta}$ is estimable because

$$(1 \;\; 0 \;\; 0) \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = 0.$$

- So is $(0 \;\; 1 \;\; 2) \; \boldsymbol{\beta}$ because

$$(0 \;\; 1 \;\; 2) \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = 0.$$

- But $(0 \;\; 1 \;\; 0) \; \boldsymbol{\beta}$ is not because

$$(0 \;\; 1 \;\; 0) \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = 2 \neq 0.$$

1.3 Fourth Method

A fourth method is to find some g-inverse of $\mathbf{X}'\mathbf{X}$, denoted by $(\mathbf{X}'\mathbf{X})^-$. Then $\mathbf{k}'\boldsymbol{\beta}$ is estimable if and only if

$$\mathbf{k}'(\mathbf{X}'\mathbf{X})^- \; \mathbf{X}'\mathbf{X} \; = \; \mathbf{k}'.$$

A definition of and methods for computing a g-inverse are presented in Chapter 3.

In the example

$$\mathbf{X}'\mathbf{X} \; = \begin{pmatrix} 4 & 7 & 14 \\ 7 & 15 & 30 \\ 14 & 30 & 60 \end{pmatrix},$$

and a g-inverse is

$$\frac{1}{11} \begin{pmatrix} 15 & -7 & 0 \\ -7 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

- $(1\ \ 0\ \ 0)\ (\mathbf{X}'\mathbf{X})^{-}\ \mathbf{X}'\mathbf{X}\ =\ (1\ \ 0\ \ 0)$. Therefore $(1\ \ 0\ \ 0)\ \boldsymbol{\beta}$ is estimable.
- $(0\ \ 1\ \ 2)\ (\mathbf{X}'\mathbf{X})^{-}\ \mathbf{X}'\mathbf{X}\ =\ (0\ \ 1\ \ 2)$. Therefore $(0\ \ 1\ \ 2)\ \boldsymbol{\beta}$ is estimable.
- $(0\ \ 1\ \ 0)\ (\mathbf{X}'\mathbf{X})^{-}\ \mathbf{X}'\mathbf{X}\ =\ (0\ \ 1\ \ 2)$. Therefore $(0\ \ 1\ \ 0)\ \boldsymbol{\beta}$ is not estimable.
- Related to this fourth method any linear function of

  $$(\mathbf{X}'\mathbf{X})^{-}\ \mathbf{X}'\mathbf{X}\boldsymbol{\beta}$$

  is estimable.

If rank $(\mathbf{X}) = p =$ the number of columns in $\mathbf{X}$, any linear function of $\boldsymbol{\beta}$ is estimable. In that case the only g-inverse of $\mathbf{X}'\mathbf{X}$ is $(\mathbf{X}'\mathbf{X})^{-1}$, a regular inverse. Then by the fourth method

$$\mathbf{k}'\ (\mathbf{X}'\mathbf{X})^{-}\ \mathbf{X}'\mathbf{X}\ =\ \mathbf{k}'(\mathbf{X}'\mathbf{X})^{-1}\ \mathbf{X}'\mathbf{X}\ =\ \mathbf{k}'\ \mathbf{I}\ =\ \mathbf{k}'.$$

Therefore, any $\mathbf{k}'\boldsymbol{\beta}$ is estimable.

There is an extensive literature on generalized inverses. See for example, Searle (1971b, 1982), Rao and Mitra (1971) and Harville(1999??).