Chapter 18
The Three Way Classification

C. R. Henderson

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This chapter deals with a 3 way classification model,

$$y_{ijkm} = \mu + a_i + b_j + c_k + ab_{ij} + ac_{ik} + bc_{jk} + abc_{ijk} + e_{ijkm}. \quad (1)$$

We need to specify the distributional properties of the elements of this model.

1 The Three Way Fixed Model

We first illustrate a fixed model with $Var(\mathbf{e}) = \mathbf{I}\sigma_e^2$. A simple way to approach this model is to write it as

$$y_{ijkm} = \mu_{ijk} + e_{ijkm}. \quad (2)$$

Then BLUE of μ_{ijk} is $\overline{y}_{ijk.}$ provided $n_{ijk} > 0$. Also BLUE of

$$\sum_i \sum_j \sum_k p_{ijk}\, \mu_{ijk} = \sum_i \sum_j \sum_k p_{ijk}\, \overline{y}_{ijk.},$$

where summation is over subclasses that are filled. But if subclasses are missing, there may not be linear functions of interest to the experimenter. Analogous to the two-way fixed model we have these definitions.

$$\begin{aligned}
a \text{ effects} &= \overline{\mu}_{i..} - \overline{\mu}_{...}, \\
b \text{ effects} &= \overline{\mu}_{.j.} - \overline{\mu}_{...}, \\
c \text{ effects} &= \overline{\mu}_{..k} - \overline{\mu}_{...}, \\
ab \text{ interactions} &= \overline{\mu}_{ij.} - \overline{\mu}_{i..} - \overline{\mu}_{.j.} + \overline{\mu}_{...}, \\
abc \text{ interactions} &= \mu_{ijk} - \overline{\mu}_{ij.} - \overline{\mu}_{i.k} - \overline{\mu}_{.jk} \\
&\quad + \overline{\mu}_{i..} + \overline{\mu}_{.j.} + \overline{\mu}_{..k} - \overline{\mu}_{...}\,. \quad (3)
\end{aligned}$$

None of these is estimable if a single subclass is missing. Consequently, the usual tests of hypotheses cannot be effected exactly.

2 The Filled Subclass Case

Suppose we wish to test the hypotheses that a effects, b effects, c effects, ab interactions, ac interactions, bc interactions, and abc interactions are all zero where these are defined as in (18.3). Three different methods will be described. The first two involve setting up least squares equations reparameterized by

$$\begin{aligned} \sum_i a_i &= \Sigma\, b_j = \Sigma\, c_k = 0 \\ \sum_j ab_{ij} &= 0 \text{ for all } i, \text{ etc.} \\ \sum_{jk} abc_{ijk} &= 0 \text{ for all } i, \text{ etc.} \end{aligned} \tag{4}$$

We illustrate this with a $2 \times 3 \times 4$ design with subclass numbers and totals as follows

	n_{ijk}											
	b_1				b_2				b_3			
a	c_1	c_2	c_3	c_4	c_1	c_2	c_3	c_4	c_1	c_2	c_3	c_4
1	3	5	2	6	5	2	1	4	5	2	1	1
2	7	2	5	1	6	2	4	3	3	4	6	1

	$y_{ijk.}$											
	b_1				b_2				b_3			
a	c_1	c_2	c_3	c_4	c_1	c_2	c_3	c_4	c_1	c_2	c_3	c_4
1	53	110	41	118	91	31	9	55	96	31	8	12
2	111	43	89	9	95	26	61	35	52	55	97	10

The first 7 columns of $\overline{\mathbf{X}}$ are

$$\begin{pmatrix}
1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & -1 & -1 & -1 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 0 & 0 \\
1 & 1 & -1 & -1 & 0 & 1 & 0 \\
1 & 1 & -1 & -1 & 0 & 0 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & 0 & 1 & 0 & 0 \\
1 & -1 & 1 & 0 & 0 & 1 & 0 \\
1 & -1 & 1 & 0 & 0 & 0 & 1 \\
1 & -1 & 1 & 0 & -1 & -1 & -1 \\
1 & -1 & 0 & 1 & 1 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 & 1 & 0 \\
1 & -1 & 0 & 1 & 0 & 0 & 1 \\
1 & -1 & 0 & 1 & -1 & -1 & -1 \\
1 & -1 & -1 & -1 & 1 & 0 & 0 \\
1 & -1 & -1 & -1 & 0 & 1 & 0 \\
1 & -1 & -1 & -1 & 0 & 0 & 1 \\
1 & -1 & -1 & -1 & -1 & -1 & -1
\end{pmatrix}.$$

The first column pertains to μ, the second to $\mathbf{a}$, the next two to $\mathbf{b}$, and the last 3 to $\mathbf{c}$. The remaining 17 columns are formed by operations on columns 2-7. Column 8 is formed by taking the products of corresponding elements of columns 2 and 3. Thus these are 1(1), 1(1), 1(1), 1(1), 1(0), …, -1(-1). The other columns are as follows: $9 = 2 \times 4$, $10 = 2 \times 5$, $11 = 2 \times 6$, $12 = 2 \times 7$, $13 = 3 \times 5$, $14 = 3 \times 6$, $15 = 3 \times 7$, $16 = 4 \times 5$, $17 = 4 \times 6$, $18 = 4 \times 7$, $19 = 2 \times 13$, $20 = 2 \times 14$, $21 = 2 \times 15$, $22 = 2 \times 16$, $23 = 2 \times 17$, $24 = 2 \times 18$.

This gives the following for columns 8-16 of $\overline{\mathbf{X}}$

$$
\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & -1 \\
-1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 \\
-1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
-1 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
-1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
-1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
-1 & 0 & 1 & 1 & 1 & -1 & -1 & -1 & 0 \\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 \\
1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & -1 \\
1 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\
1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
$$

and for columns 17-24 of $\overline{\mathbf{X}}$,

$$\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & -1 & 0 & 0 & 0 & -1 & -1 & -1 \\
0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & -1
\end{pmatrix}.$$

Then the least squares coefficient matrix is $\overline{\mathbf{X}}'\mathbf{N}\overline{\mathbf{X}}$, where $\mathbf{N}$ is a diagonal matrix of n_{ijk}. The right hand sides are $\overline{\mathbf{X}}'\mathbf{y}.$, where $\mathbf{y}.$ is the vector of subclass totals. The coefficient matrix of the equations is in (18.5) ... (18.7). The right hand side is (1338, -28, 213, 42, 259, 57, 66, 137, 36, -149, -83, -320, -89, -38, -80, -30, -97, -103, -209, -16, -66, -66, 11, 19)'.

Upper left 12 × 12

$$
\begin{pmatrix}
81 & -7 & 8 & 4 & 13 & 1 & 3 & 6 & 2 & -9 & -5 & -17 \\
 & 81 & 6 & 2 & -9 & -5 & -17 & 8 & 4 & 13 & 1 & 3 \\
 & & 54 & 23 & -3 & -4 & -5 & -4 & -5 & -11 & 0 & -3 \\
 & & & 50 & -2 & -7 & -7 & -5 & -8 & -4 & 1 & 1 \\
 & & & & 45 & 16 & 16 & -11 & -4 & 3 & 6 & 6 \\
 & & & & & 33 & 16 & 0 & 1 & 6 & 7 & 6 \\
 & & & & & & 35 & -3 & 1 & 6 & 6 & -5 \\
 & & & & & & & 54 & 23 & -3 & -4 & -5 \\
 & & & & & & & & 50 & -2 & -7 & -7 \\
 & & & & & & & & & 45 & 16 & 16 \\
 & & & & & & & & & & 33 & 16 \\
 & & & & & & & & & & & 35
\end{pmatrix}. \tag{5}
$$

Upper right 12 × 12 and (lower left 12 × 12)'

$$
\begin{pmatrix}
-3 & -4 & -5 & -2 & -7 & -7 & -11 & 0 & -3 & -4 & 1 & 1 \\
-11 & 0 & -3 & -4 & 1 & 1 & -3 & -4 & -5 & -2 & -7 & -7 \\
9 & 4 & 5 & 6 & 4 & 5 & -7 & -4 & -13 & 2 & -2 & -5 \\
6 & 4 & 5 & 10 & 1 & 3 & 2 & -2 & -5 & 0 & -3 & -9 \\
7 & 5 & 5 & 8 & 5 & 5 & -1 & 5 & 5 & -2 & 1 & 1 \\
5 & 6 & 5 & 5 & 3 & 5 & 5 & 10 & 5 & 1 & 3 & 1 \\
5 & 5 & 5 & 5 & 5 & 3 & 5 & 5 & 7 & 1 & 1 & 3 \\
-7 & -4 & -13 & 2 & -2 & -5 & 9 & 4 & 5 & 6 & 4 & 5 \\
2 & -2 & -5 & 0 & -3 & -9 & 6 & 4 & 5 & 10 & 1 & 3 \\
-1 & 5 & 5 & -2 & 1 & 1 & 7 & 5 & 5 & 8 & 5 & 5 \\
5 & 10 & 5 & 1 & 3 & 1 & 5 & 6 & 5 & 5 & 3 & 5 \\
5 & 5 & 7 & 1 & 1 & 3 & 5 & 5 & 5 & 5 & 5 & 3
\end{pmatrix}. \tag{6}
$$

Lower right 12 × 12

$$
\begin{pmatrix}
27 & 9 & 9 & 10 & 2 & 2 & 3 & 5 & 5 & 2 & 0 & 0 \\
 & 22 & 9 & 2 & 8 & 2 & 5 & 6 & 5 & 0 & -2 & 0 \\
 & & 23 & 2 & 2 & 9 & 5 & 5 & -3 & 0 & 0 & -5 \\
 & & & 28 & 9 & 9 & 2 & 0 & 0 & 2 & 1 & 1 \\
 & & & & 19 & 9 & 0 & -2 & 0 & 1 & -1 & 1 \\
 & & & & & 21 & 0 & 0 & -5 & 1 & 1 & -7 \\
 & & & & & & 27 & 9 & 9 & 10 & 2 & 2 \\
 & & & & & & & 22 & 9 & 2 & 8 & 2 \\
 & & & & & & & & 23 & 2 & 2 & 9 \\
 & & & & & & & & & 28 & 9 & 9 \\
 & & & & & & & & & & 19 & 9 \\
 & & & & & & & & & & & 21
\end{pmatrix}. \tag{7}
$$

The resulting solution is (15.3392, .5761, 2.6596, -1.3142, 2.0092, 1.5358, -.8864, 1.3834, -.4886, .4311, .2156, -2.5289, -3.2461, 2.2154, 2.0376, .9824, -1.3108, -1.0136, -1.4858, -1.9251, 1.9193, .6648, .9469, -6836).

One method for finding the numerator sums of squares is to compare reductions, that is, subtracting the reduction when each factor and interaction is deleted from the reduction under the full model. For A, equation and unknown 2 is deleted, for B equations 3 and 4 are deleted, ..., for ABC equations 19-24 are deleted. The reduction under the full model is 22879.49 which is also simply

$$\sum_i \sum_j \sum_k y_{ijk.}^2 / n_{ijk}.$$

The sums of squares with their d.f. are as follows.

	d.f.	SS
A	1	17.88
B	2	207.44
C	3	192.20
AB	2	55.79
AC	3	113.25
BC	6	210.45
ABC	6	92.73

The denominator MS to use is $\hat{\sigma}_e^2 = (\mathbf{y}'\mathbf{y}$ - reduction in full model)/(81-24), where 81 is n, and 24 is the rank of the full model coefficient matrix.

A second method, usually easier, is to compute for the numerator

$$\text{SS} = (\boldsymbol{\beta}_i^o)'(Var(\boldsymbol{\beta}_i^o))^{-1}\boldsymbol{\beta}_i^o\sigma_e^2. \quad (8)$$

$\boldsymbol{\beta}_i^o$ is a subvector of the solution, β_2^o for A; β_3^o, β_4^o for B, ..., β_{17}^o, ..., β_{24}^o for ABC. $Var(\boldsymbol{\beta}_i^o)$ is the corresponding diagonal block of the inverse of the 24 $\times$ 24 coefficient matrix, not shown, multiplied by σ_e^2 . Thus

$$\text{SS for } A = .5761 \ (.0186)^{-1} \ .5761,$$

$$\text{SS for } B = (2.6596 \quad -1.3142) \begin{pmatrix} .0344 & -.0140 \\ -.0140 & .0352 \end{pmatrix}^{-1} \begin{pmatrix} 2.6596 \\ -1.3142 \end{pmatrix},$$

etc. The terms inverted are diagonal blocks of the inverse of the coefficient matrix. These give the same results as by the first method.

The third method is to compute

$$(\mathbf{K}_i'\hat{\boldsymbol{\mu}})' \left(Var(\mathbf{K}_i'\hat{\boldsymbol{\mu}})\right)^{-1} \mathbf{K}'\hat{\boldsymbol{\mu}}\sigma_e^2. \quad (9)$$

$\mathbf{K}_i'\boldsymbol{\mu} = 0$ is the hypothesis tested for the i^{th} SS. $\hat{\boldsymbol{\mu}}$ is BLUE of $\boldsymbol{\mu}$, the vector of μ_{ijk}, and this is the vector of $\overline{y}_{ijk}$.

$\mathbf{K}_A$ is the 2nd column of $\overline{\mathbf{X}}$.

$\mathbf{K}_B$ is columns 3 and 4 of $\overline{\mathbf{X}}$.

$\vdots$

$\mathbf{K}_{ABC}$ is the last 6 columns of $\overline{\mathbf{X}}$.

For example, $\mathbf{K}_B'$ for SSB is

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} & -\mathbf{1} & \mathbf{1} & \mathbf{0} & -\mathbf{1} \\ \mathbf{0} & \mathbf{1} & -\mathbf{1} & -\mathbf{1} & \mathbf{1} & -\mathbf{1} \end{pmatrix},$$

where $\mathbf{1} = (1\ 1\ 1\ 1)$ and $\mathbf{0} = (0\ 0\ 0\ 0)$.

$$Var(\hat{\boldsymbol{\mu}})/\sigma_e^2 = \mathbf{N}^{-1},$$

where $\mathbf{N}$ is the diagonal matrix of n_{ijk}. Then

$$Var(\mathbf{K}'\hat{\boldsymbol{\mu}})^{-1}\sigma_e^2 = (\mathbf{K}'\mathbf{N}^{-1}\mathbf{K})^{-1}.$$

This method leads to the same sums of squares as the other 2 methods.

3 Missing Subclasses In The Fixed Model

When one or more subclasses is missing, the usual estimates and tests of main effects and interactions cannot be made. If one is satisfied with estimating and testing functions like $\mathbf{K}'\boldsymbol{\mu}$, where $\boldsymbol{\mu}$ is the vector of μ_{ijk} corresponding to filled subclasses, BLUE and exact tests are straightforward. BLUE of

$$\mathbf{K}'\boldsymbol{\mu} = \mathbf{K}'\overline{\mathbf{y}}, \tag{10}$$

where $\overline{\mathbf{y}}.$ is the vector of means of filled 3 way subclasses. The numerator SS for testing the hypothesis that $\mathbf{K}'\boldsymbol{\mu} = \mathbf{c}$ is

$$(\mathbf{K}'\overline{\mathbf{y}}. - \mathbf{c})' \ Var(\mathbf{K}'\overline{\mathbf{y}}.)^{-1}(\mathbf{K}'\overline{\mathbf{y}}. - \mathbf{c})\sigma_e^2. \tag{11}$$

$$Var(\mathbf{K}'\overline{\mathbf{y}}.)/\sigma_e^2 = \mathbf{K}'\mathbf{N}^{-1}\mathbf{K}, \tag{12}$$

where $\mathbf{N}$ is a diagonal matrix of subclass numbers. The statistic of (18.11) is distributed as central $\chi^2\sigma_e^2$ with d.f. equal to the number of linearly independent rows of $\mathbf{K}'$. Then the corresponding MS divided by $\hat{\sigma}_e^2$ is distributed as F under the null hypothesis.

Unfortunately, if many subclasses are missing, the experimenter may have difficulty in finding functions of interest to estimate and test. Most of them wish correctly or otherwise to find estimates and tests that mimic the filled subclass case. Clearly this is possible only if one is prepared to use biased estimators and approximate tests of the functions whose estimators are biased.

We illustrate some biased methods with the following $2 \times 3 \times 4$ example.

	n_{ijk}											
	b_1				b_2				b_3			
a	c_1	c_2	c_3	c_4	c_1	c_2	c_3	c_4	c_1	c_2	c_3	c_4
1	3	5	2	6	5	2	0	4	5	2	0	0
2	7	2	5	0	6	2	4	3	3	4	6	0

	$y_{ijk.}$											
	b_1				b_2				b_3			
a	c_1	c_2	c_3	c_4	c_1	c_2	c_3	c_4	c_1	c_2	c_3	c_4
1	53	110	41	118	91	31	–	55	96	31	–	–
2	111	43	89	–	95	26	61	35	52	55	97	–

Note that 5 of the potential 24 *abc* subclasses are empty and one of the potential 12 *bc* subclasses is empty. All *ab* and *ac* subclasses are filled. Some common procedures are

1. Estimate and test main effects pretending that no interactions exist.

2. Estimate and test main effects, *ac* interactions, and *bc* interactions pretending that *bc* and *abc* interactions do not exist.

3. Estimate and test under a model in which interactions sum to 0 and in which each of the 5 missing *abc* and the one missing *bc* interactions are assumed $= 0$.

All of these clearly are biased methods, and their "goodness" depends upon the closeness of the assumptions to the truth. If one is prepared to use biased estimators, it seems more logical to me to attempt to minimize mean squared errors by using prior values for average sums of squares and products of interactions. Some possibilities for our example are:

1. Priors on **abc** and **bc**, the interactions associated with missing subclasses.

2. Priors on all interactions.

3. Priors on all interactions and on all main effects.

Obviously there are many other possibilities, e.g. priors on **c** and all interactions.

The first method above might have the greatest appeal since it results in biases due only to **bc** and **abc** interactions. No method for estimating main effects exists that does not contain biases due to these. But the first method does avoid biases due to main effects, **ab**, and **ac** interactions. This method will be illustrated. Let μ, **a**, **b**, **c**, **ab**, **ac** be treated as fixed. Consequently we have much confounding among them. The rank of the submatrix of $\mathbf{X}'\mathbf{X}$ pertaining to them is 1 + (2-1) + (3-1) + (4-1) + (2-1)(3-1) + (2-1)(4-1) = 12. We set up least squares equations with **ab**, **ac**, **bc**, and **abc** including missing subclasses for **bc** and **abc**. The submatrix for **ab** and **ac** has order, 14 and rank, 12. Treating **bc** and **abc** as random results in a mixed model coefficient matrix with order 50, and rank 48. The OLS coefficient matrix is in (18.13) to (18.18). The upper 26 $\times$ 26 block is in (18.13) to (18.15), the upper right 26 $\times$ 24 block is in (18.16) to (18.17), and the lower 24 $\times$ 24 block is in (18.18).

$$\begin{pmatrix}
16 & 0 & 0 & 0 & 0 & 0 & 3 & 5 & 2 & 6 & 0 & 0 & 0 \\
 & 11 & 0 & 0 & 0 & 0 & 5 & 2 & 0 & 4 & 0 & 0 & 0 \\
 & & 7 & 0 & 0 & 0 & 5 & 2 & 0 & 0 & 0 & 0 & 0 \\
 & & & 14 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 2 & 5 \\
 & & & & 15 & 0 & 0 & 0 & 0 & 0 & 6 & 2 & 4 \\
 & & & & & 13 & 0 & 0 & 0 & 0 & 3 & 4 & 6 \\
 & & & & & & 13 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & 9 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & 2 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & 10 & 0 & 0 & 0 \\
 & & & & & & & & & & 16 & 0 & 0 \\
 & & & & & & & & & & & 8 & 0 \\
 & & & & & & & & & & & & 15
\end{pmatrix} \tag{13}$$

$$\begin{pmatrix}
0 & 3 & 5 & 2 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 5 & 2 & 0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 2 & 0 & 0 \\
0 & 7 & 2 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 6 & 2 & 4 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 4 & 6 & 0 \\
0 & 3 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\
0 & 0 & 5 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\
0 & 7 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 5 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 6 & 0
\end{pmatrix} \tag{14}$$

$$
\begin{pmatrix}
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\
 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & 11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & 4 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & 7 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & 8 & 0 & 0 & 0 \\
 & & & & & & & & & & 6 & 0 & 0 \\
 & & & & & & & & & & & 6 & 0 \\
 & & & & & & & & & & & & 0
\end{pmatrix} \tag{15}
$$

$$
\begin{pmatrix}
3 & 5 & 2 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 5 & 2 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\
0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \tag{16}
$$

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & 2 & 4 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 4 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 5 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \qquad (17)
$$

$$
\mathrm{diag}(3, 5, 2, 6, 5, 2, 0, 4, 5, 2, 0, 0, 7, 2, 5, 0, 6, 2, 4, 3, 3, 4, 6, 0). \qquad (18)
$$

The right hand side is

$$
\begin{array}{rrrrrrrrrrr}
[322, & 177, & 127, & 243, & 217, & 204, & 240, & 172, & 41, & 173, & 258, \\
124, & 247, & 35, & 164, & 153, & 130, & 118, & 186, & 57, & 61, & 90, \\
148, & 86, & 97, & 0, & 53, & 110, & 41, & 118, & 91, & 31, & 0, \\
55, & 96, & 31, & 0, & 0, & 111, & 43, & 89, & 0, & 95, & 26, \\
61, & 35, & 52, & 55, & 97, & 0] & & & & &
\end{array}
$$

We use the diagonalization method and assume that the pseudo-variances are $\sigma^2_{bc} = .3\ \sigma^2_e$, $\sigma^2_{abc} = .6\ \sigma^2_e$. Accordingly we add $.3^{-1}$ to the 15-26 diagonals and $.6^{-1}$ to the 27-50 diagonals of the OLS equations. This gives the following solution

$$
\begin{aligned}
\mathbf{ab} &= (20.664,\ 16.724,\ 17.812,\ 0,\ -3.507,\ -2.487)' \\
\mathbf{ac} &= (.047,\ -.618,\ 0,\ -1.949,\ 18.268,\ 17.976,\ 18.401,\ 15.441)' \\
\mathbf{bc} &= (-1.132,\ 1.028,\ -.164,\ .268,\ .541,\ -.366,\ .093,\ -.268, \\
&\qquad .591,\ -.662,\ .071,\ 0)'
\end{aligned}
$$

$$\begin{aligned}\mathbf{abc} &= (-1.229,\ .694,\ 0,\ .535,\ .666,\ -.131,\ 0,\ -.535,\ .563,\\ &\quad -.563,\ 0,\ 0,\ -1.034,\ 1.362,\ -.328,\ 0,\ .416,\ -.601,\\ &\quad .186,\ 0,\ .618,\ -.760,\ .142,\ 0)'.\end{aligned}$$

The biased estimator of μ_{ijk} is $ab^o_{ij} + ac^o_{ik} + bc^o_{jk} + abc^o_{ijk}$. These are in tabular form ordered c in b in a by rows.

$$\mathbf{K} = 8^{-1}\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ -1 & -1 \\ -1 & -1 \\ -1 & -1 \\ -1 & -1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ -1 & -1 \\ -1 & -1 \\ -1 & -1 \\ -1 & -1 \end{pmatrix}, \quad \hat{\boldsymbol{\mu}} = \begin{pmatrix} 18.35 \\ 21.77 \\ 20.50 \\ 19.52 \\ 17.98 \\ 15.61 \\ 16.82 \\ 13.97 \\ 19.01 \\ 15.97 \\ 17.88 \\ 15.86 \\ 16.10 \\ 20.37 \\ 17.91 \\ 15.71 \\ 15.72 \\ 13.50 \\ 15.17 \\ 11.67 \\ 16.99 \\ 14.07 \\ 16.13 \\ 12.95 \end{pmatrix}$$

The variance-covariance matrix of these $\hat{\mu}_{ijk}$ is $\overline{\mathbf{X}}\mathbf{C}\overline{\mathbf{X}}'\sigma^2_e$, where $\overline{\mathbf{X}}$ is the 24 x 50 incidence matrix for $\overline{y}_{ijk}$, and $\mathbf{C}$ is a g-inverse of the mixed model coefficient matrix. Approximate tests of hypotheses of $\mathbf{K}'\boldsymbol{\mu} = \mathbf{c}$ can be effected by computing

$$(\mathbf{K}'\hat{\boldsymbol{\mu}} - \mathbf{c})'[\mathbf{K}'\overline{\mathbf{X}}\mathbf{C}\overline{\mathbf{X}}'\mathbf{K}]^{-1}(\mathbf{K}'\hat{\boldsymbol{\mu}} - \mathbf{c})/(\text{rank }(\mathbf{K}'\overline{\mathbf{X}})\hat{\sigma}^2_e).$$

Under the null hypothesis this is distributed approximately as F.

To illustrate suppose we wish to test that all $\overline{\mu}_{.j.}$ are equal. $\mathbf{K}'$ and $\hat{\boldsymbol{\mu}}$ for this test are shown above and $\mathbf{c} = \mathbf{0}$. $\mathbf{K}'\hat{\boldsymbol{\mu}} = (2.66966 \quad -1.05379)'$. The pseudo-variances, σ^2_{bc} and σ^2_{abc}, could be estimated quite easily by Method 3. One could estimate σ^2_e by $\mathbf{y}'\mathbf{y}$ - reduction under full model, and this is simply

$$\mathbf{y}'\mathbf{y} - \sum_i \sum_j \sum_k y^2_{ijk.}/n_{ijk}.$$

Then we divide by n - the number of filled subclasses. Three reductions are needed to estimate σ^2_{bc} and σ^2_{abc}. The easiest ones are probably

Red (full model) described above.

Red (ab,ac,bc).

Red (ab,ac).

Partition the OLS coefficient matrix as

$$(\mathbf{C}_1 \quad \mathbf{C}_2 \quad \mathbf{C}_3).$$

$\mathbf{C}_1$ represents the first 14 cols., $\mathbf{C}_2$ the next 12, and $\mathbf{C}_3$ the last 24. Then compute $\mathbf{C}_2\mathbf{C}_2'$ and $\mathbf{C}_3\mathbf{C}_3'$. Let $\mathbf{Q}_2$ be the g-inverse of the matrix for Red (ab,ac,bc), which is the LS coefficient matrix with rows (and cols.) 27-50 set to 0. $\mathbf{Q}_3$ is the g-inverse for Red (ab,ac), which is the LS coefficient matrix with rows (and cols.) 15-50 set to 0. Then

$$\begin{aligned} E[\text{Red (full)}] &= 19\sigma^2_e + n(\sigma^2_{bc} + \sigma^2_{abc}) + t, \\ E[\text{Red } (ab, ac, bc)] &= 17\sigma^2_e + n\sigma^2_{bc} + tr\mathbf{Q}_2\mathbf{C}_3\mathbf{C}_3'\sigma^2_{abc} + t, \\ E[\text{Red } (ab, ac) &= 12\sigma^2_e + tr\mathbf{Q}_3\mathbf{C}_2\mathbf{C}_2'\sigma^2_{bc} + tr\mathbf{Q}_3\mathbf{C}_3\mathbf{C}_3'\sigma^2_{abc} + t. \end{aligned}$$

t is a quadratic in the fixed effects. The coefficient of σ^2_e is in each case the rank of the coefficient matrix used in the reduction.

4 The Three Way Mixed Model

Mixed models could be of two general types, namely one factor fixed and two random such as **a** fixed and **b** and **c** random, or with two factors fixed and one factor random, e.g. **a** and **b** fixed with **c** random. In either of these we would need to consider whether the populations are finite or infinite and whether the elements are related in any way. With **a** and **b** fixed and **c** random we would have fixed **ab** interaction and random **ac**, **bc**, **abc** interactions. With **a** fixed and **b** and **c** random all interactions would be random.

We also need to be careful about what we can estimate and predict. With **a** fixed and **b** and **c** random we can predict elements of **ab**, **ac**, and **abc** only for the levels of **a** in the experiment. With **a** and **b** fixed we can predict elements of **ac**, **bc**, **abc** only for the levels of both **a** and **b** in the experiment. For infinite populations of **b** and **c** in the first case and **c** in the second we can predict for levels of **b** and **c** (or **c**) outside the experiment. BLUP of them is 0. Thus in the case with **c** random, **a** and **b** fixed, BLUP of the 1,2,20 subclass when the number of levels of **c** in the experiment <20, is

$$\mu^o + a^o_1 + b^o_2 + ab^o_{12}.$$

In contrast, if the number of levels of **c** in the experiment >19, BLUP is

$$\mu^o + a_1^o + b_2^o + \hat{c}_{20} + \hat{a}c_{1,20} + \hat{b}c_{2,20} + ab_{12}^o + a\hat{b}c_{12,20}.$$

In the case with $\mathbf{a}, \mathbf{b}$ fixed and $\mathbf{c}$ random, we might choose to place a prior on **ab**, especially if **ab** subclasses are missing in the data. The easiest way to do this would be to treat **ab** as a pseudo random variable with variance $= \mathbf{I}\sigma^2_{ab}$, which could be estimated. We could also use priors on **a** and **b** if we choose, and then the mixed model equations would mimic the 3 way random model.