Chapter 16
The One-Way Classification

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This and subsequent chapters will illustrate principles of Chapter 1-15 as applied to specific designs and classification of data. This chapter is concerned with a model,

$$y_{ij} = \mu + a_i + e_{ij}. \tag{1}$$

Thus data can be classified with n_i observations on the i^{th} class and with the total of observations in that class $= y_{i.}$. Now (1) is not really a model until we specify what population or populations were sampled and what are the properties of these populations. One possibility is that in conceptual repeated sampling μ and a_i always have the same values, and the e_{ij} are random samples from an infinite population of uncorrelated variables with mean 0, and common variance, σ_e^2. That is, the variance of the population of $\mathbf{e}$ is $\mathbf{I}\sigma_e^2$, and the sample vector of n elements has expectation null and variance $= \mathbf{I}\sigma_e^2$. Note that $\text{Var}(e_{ij})$ is assumed equal to $\text{Var}(e_{i'j})$, $i \neq i'$.

1 Estimation and Tests For Fixed a

Estimation and tests of hypothesis are simple under this model. The mixed model equations are OLS equations since $\mathbf{Zu}$ does not exist and since $Var(\mathbf{e}) = \mathbf{I}\sigma_e^2$. They are

$$\frac{1}{\sigma_e^2} \begin{pmatrix} n_. & n_1 & n_2 & \dots \\ n_1 & n_1 & 0 & \dots \\ n_2 & 0 & n_2 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} \mu^o \\ a_1^o \\ a_2^o \\ \vdots \end{pmatrix} = \begin{pmatrix} y_{..} \\ y_{1.} \\ y_{2.} \\ \vdots \end{pmatrix} \frac{1}{\sigma_e^2}. \tag{2}$$

The $\mathbf{X}$ matrix has $t+1$ columns, where $t =$ the number of levels of a, but the rank is t. None of the elements of the model is estimable. We can estimate

$$\mu + \sum_{i=1}^{t} k_i a_i,$$

where

$$\sum_i k_i = 1,$$

or

$$\sum_{i}^{t} k_i a_i,$$

if

$$\sum_{i} k_i = 0.$$

For example $\mu + a_i$ is estimable, $a_i - a_{i'}$ is estimable, and

$$a_1 - \sum_{i=2}^{t} k_i a_i,$$

with

$$\sum_{i=2}^{t} k_i = 1,$$

is estimable. The simplest solution to (2) is $\mu^o = 0$, $a_i^o = \overline{y}_{i.}$. This solution corresponds to the following g-inverse.

$$\begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & n_1^{-1} & 0 & \cdots \\ 0 & 0 & n_2^{-1} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

Let us illustrate with the following example

$$\begin{aligned} (n_1,\ n_2,\ n_3) &= (8, 3, 4), \\ (y_{1.},\ y_{2.},\ y_{3.}) &= (49, 16, 13), \\ \mathbf{y}'\mathbf{y} &= 468. \end{aligned}$$

The OLS equations are

$$\frac{1}{\sigma_e^2} \begin{pmatrix} 15 & 8 & 3 & 4 \\ & 8 & 0 & 0 \\ & & 3 & 0 \\ & & & 4 \end{pmatrix} \begin{pmatrix} \mu^o \\ a_1^o \\ a_2^o \\ a_3^o \end{pmatrix} = \begin{pmatrix} 78 \\ 49 \\ 16 \\ 13 \end{pmatrix} \frac{1}{\sigma_e^2}. \tag{3}$$

A solution is (0, 49/8, 16/3, 13/4). The corresponding g-inverse of the coefficient matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ & 8^{-1} & 0 & 0 \\ & & 3^{-1} & 0 \\ & & & 4^{-1} \end{pmatrix} \sigma_e^2.$$

Suppose one wishes to estimate $a_1 - a_2$, $a_1 - a_3$, $a_2 - a_3$. Then from the above solution these would be $\frac{49}{8} - \frac{16}{3}$, $\frac{49}{8} - \frac{13}{4}$, $\frac{16}{3} - \frac{13}{4}$. The variance-covariance matrix of these estimators is

$$\begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 8^{-1} & 0 & 0 \\ 0 & 0 & 3^{-1} & 0 \\ 0 & 0 & 0 & 4^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix} \sigma_e^2 . \tag{4}$$

We do not know σ_e^2 but it can be estimated easily by

$$\begin{aligned} \hat{\sigma}_e^2 &= (\mathbf{y}'\mathbf{y} - \sum_i y_{i.}^2/n_i)/(15-3) \\ &= (468 \; - \; 427.708)/12 \\ &= 3.36. \end{aligned}$$

Then we can substitute this for σ_e^2 to obtain estimated sampling variances.

Suppose we want to test the hypothesis that the levels of a_i are equal. This can be expressed as a test that

$$\begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \mu \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

$$Var(\mathbf{K}'\beta^o) = \mathbf{K}'(\text{g} - \text{inverse})\mathbf{K} = \begin{pmatrix} .45833 & .125 \\ .125 & .375 \end{pmatrix} \sigma_e^2$$

with

$$\text{inverse} = \begin{pmatrix} 2.4 & -.8 \\ -.8 & 2.9333 \end{pmatrix} \frac{1}{\sigma_e^2} .$$

$$\mathbf{K}'\boldsymbol{\beta}^o = (.79167 \;\; 2.875)'.$$

Then

$$\begin{aligned} \text{numerator SS} &= (.79167 \;\; 2.875) \begin{pmatrix} 2.4 & -.8 \\ & 2.9333 \end{pmatrix} \begin{pmatrix} .79167 \\ 2.875 \end{pmatrix} \\ &= 22.108. \end{aligned}$$

The same numerator can be computed from

$$\sum_i \frac{y_{i.}^2}{n_i} - \frac{y_{..}^2}{n_.} = 427.708 \; - \; 405.6 = 22.108.$$

Then the test that a_i are equal is $\frac{22.108/2}{3.36}$ which is distributed as $F_{2,12}$ under the null hypothesis.

2 Levels of a Equally Spaced

In some experiments the levels of **a** (treatments) are chosen to be "equally spaced". For example, if treatments are percent protein in the diet, the levels chosen might be 10%, 12%, 14%, 16%, 18%. Suppose we have 5 such treatments with n_i = (5,2,1,3,8) and $y_{i.}$ = (10,7,3,8,33). Let the full model be

$$y_{ij} = \mu + \beta_1 x_i + \beta_2 x_i^2 + \beta_3 x_i^3 + \beta_4 x_i^4 + e_{ij} \tag{5}$$

where x_i = (1,2,3,4,5). With Var(e) = $\mathbf{I}\sigma^2$ the OLS equations under the full model are

$$\begin{pmatrix} 19 & 64 & 270 & 1240 & 5886 \\ & 270 & 1240 & 5886 & 28,384 \\ & & 5886 & 28,384 & 138,150 \\ & & & 138,150 & 676,600 \\ & & & & 3,328,686 \end{pmatrix} \begin{pmatrix} \hat{\mu} \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \\ \hat{\beta}_4 \end{pmatrix} = \begin{pmatrix} 61 \\ 230 \\ 1018 \\ 4784 \\ 23,038 \end{pmatrix}. \tag{6}$$

The solution is [-4.20833, 9.60069, -3.95660, .58681, -.02257]. The reduction in SS is 210.958 which is exactly the same as $\sum_i y_{i.}^2/n_i$. A common set of tests is the following.

$\beta_1 = 0$ assuming β_2, β_3, β_4 non-existent.

$\beta_2 = 0$ assuming β_3, β_4 non-existent.

$\beta_3 = 0$ assuming β_4 non-existent.

$\beta_4 = 0$.

This can be done by computing the following reductions.

1. Red (full model).

2. Red (μ, β_1, β_2, β_3).

3. Red (μ, β_1, β_2).

4. Red (μ, β_1).

5. Red (μ).

Then the numerators for tests above are reductions 4-5, 3-4, 2-3, 1-2 respectively.

Red (2) is obtained by dropping the last equation of (6). This gives the solution (-3.2507, 7.7707, -2.8325, .3147) with reduction = 210.952. The other reductions by successive dropping of an equation are 207.011, 206.896, 195.842. This leads to mean squares each with 1 df.

Linear	11.054
Quadratic	.115
Cubic	3.941
Quartic	.006

The sum of these is equal to the reduction under the full model minus the reduction due to μ alone.

3 Biased Estimation of $\mu + a_i$

Now we consider biased estimation under the assumption that values of $\mathbf{a}$ are unpatterned. Using the same data as in the previous section we assume for purposes of illustration that $Var(\mathbf{e}) = \frac{5}{6}\mathbf{I}$, and that the average values of squares and products of the deviations of $\mathbf{a}$ from $\bar{\mathbf{a}}$ are

$$\frac{1}{8}\begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ & 4 & -1 & -1 & -1 \\ & & 4 & -1 & -1 \\ & & & 4 & -1 \\ & & & & 4 \end{pmatrix}. \tag{7}$$

Then the equations for minimum mean squared error estimation are

$$\begin{pmatrix} 22.8 & 6.0 & 2.4 & 1.2 & 3.6 & 9.6 \\ .9 & 4.0 & -.3 & -.15 & -.45 & -1.2 \\ -1.35 & -.75 & 2.2 & -.15 & -.45 & -1.2 \\ -2.1 & -.75 & -.3 & 1.6 & -.45 & -1.2 \\ -.6 & -.75 & -.3 & -.15 & 2.8 & -1.2 \\ 3.15 & -.75 & -.3 & -.15 & -.45 & 5.8 \end{pmatrix} (\boldsymbol{\beta}^o) = \begin{pmatrix} 73.2 \\ -1.65 \\ -3.9 \\ -6.9 \\ -3.15 \\ 15.6 \end{pmatrix}. \tag{8}$$

The solution is (3.072, -.847, .257, -.031, -.281, .902). Note that

$$\sum_i \hat{a}_i = 0.$$

The estimates of differences between a_i are

	2	3	4	5
1	-1.103	-.816	-.566	-1.749
2		.288	.538	-.645
3			.250	-.933
4				-1.183

Contrast these with the corresponding BLUE. These are

	2	3	4	5
1	-1.5	-1.0	-.667	-2.125
2		.5	.833	-.625
3			.333	-1.125
4				-1.458

Generally the absolute differences are larger for BLUE.

The mean squared error of these differences, assuming that σ_e^2 and products of deviations of **a** are correct, are obtained from a g-inverse post-multiplied by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & -1 & -1 & -1 & -1 \\ 0 & -1 & 4 & -1 & -1 & -1 \\ 0 & -1 & -1 & 4 & -1 & -1 \\ 0 & -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & -1 & 4 \end{pmatrix}.$$

These are

	2	3	4	5
1	.388	.513	.326	.222
2		.613	.444	.352
3			.562	.480
4				.287

The corresponding values for BLUE are

	2	3	4	5
1	.7	1.2	.533	.325
2		1.5	.833	.625
3			1.333	1.125
4				.458

If the priors used are really correct, the MSE for biased estimators of differences are considerably smaller than BLUE.

The same biased estimators can be obtained by use of a diagonal **P**, namely .625**I**, where

$$.625 = \frac{5}{5-1}\,(.5).$$

This gives the same solution vector, but the inverse elements are different. However, mean squared errors of estimable functions such as the $\hat{a}_i - \hat{a}_j$ and $\hat{\mu} + \overline{a}$ yield the same results when applied to the inverse.

4 Model with Linear Trend of Fixed Levels of a

Assume now the same data as section 2 and that the model is

$$y_{ij} = \mu + \beta x_i + a_i + e_{ij} \tag{9}$$

where x_i = (1,2,3,4,5). Suppose that the levels of **a** are assumed to have no pattern and we use a prior value on their squares and products =

$$\begin{pmatrix} .2 & & -.05 \\ & \ddots & \\ -.05 & & 2 \end{pmatrix}.$$

Assume as before Var(**e**) $= \frac{5}{6}\mathbf{I}$. Then the equations to solve are

$$\begin{pmatrix} 22.8 & 76.8 & 6. & 2.4 & 1.2 & 3.6 & 9.6 \\ 76.8 & 324 & 6 & 4.8 & 3.6 & 14.4 & 48 \\ .36 & -2.34 & 2.2 & -.12 & -.06 & -.18 & -.48 \\ -.54 & -2.64 & -.3 & 1.48 & -.06 & -.18 & -.48 \\ -.84 & -2.94 & -.3 & -.12 & 1.24 & -.18 & -.48 \\ -.24 & -.24 & -.3 & -.12 & -.06 & 1.72 & -.48 \\ 1.26 & 8.16 & -.3 & -.12 & -.06 & -.18 & 2.92 \end{pmatrix} \begin{pmatrix} \hat{\mu} \\ \hat{\beta} \\ \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \\ \hat{a}_4 \\ \hat{a}_5 \end{pmatrix} = \begin{pmatrix} 73.2 \\ 276 \\ -.66 \\ -1.56 \\ -2.76 \\ -1.26 \\ 6.24 \end{pmatrix}. \tag{10}$$

The solution is [1.841, .400, -.145, .322, -.010, -.367, .200]. Note that

$$\sum_i \hat{a}_i = 0.$$

We need to observe precautions in interpreting the solution. β is not estimable and neither is $\mu + a_i$ nor $a_i - a_{i'}$.

We can only estimate treatment means associated with the particular level of x_i in the experiment. Thus we can estimate $\mu + a_i + x_i\beta$ where x_i = 1,2,3,4,5 for the 5 treatments respectively. The biased estimates of treatment means are

1. 1.841 + .400 - .145 = 2.096

2. 1.841 + .800 + .322 = 2.963

3. 1.841 + 1.200 - .010 = 3.031

4. 1.841 + 1.600 - .367 = 3.074

5. 1.841 + 2.000 + .200 = 4.041

The corresponding BLUE are the treatment means, (2.0, 3.5, 3.0, 2.667, 4.125).

If the true ratio of squares and products of a_i to σ_e^2 are as assumed above, the biased estimators have minimum mean squared error. Note that $E(\hat{\mu} + \hat{a}_i + x_i\hat{\beta})$ for the biased estimator is $\mu + x_i\beta +$ some function of **a** (not equal to a_i). The BLUE estimator has, of course, expectation, $\mu + x_i\beta + a_i$, that is, it is unbiased.

5 The Usual One Way Covariate Model

If, in contrast to x_i being constant for every observation on the i^{th} treatment as in Section 4, we have the more traditional covariate model,

$$y_{ij} = \mu + \beta x_{ij} + a_i + e_{ij}, \tag{11}$$

we can then estimate $\mu + a_i$ unbiasedly as well as $a_i - a_{i'}$. Again, however, if we think the a_i are unpatterned and we have some good prior value of their products, we can obtain smaller mean squared errors by using the biased method.

Now we need to consider the meaning of an estimator of $\mu + a_i$. This really is an estimator of treatment mean in hypothetical repeated sampling in which $\overline{x}_{i.} = 0$. What if the range of the x_{ij} is 5 to 21 in the sample? Can we infer from this that the the response to levels of x is that same linear function for a range of x_{ij} as low as 0? Strictly speaking we can draw inferences only for the values of x in the experiment. With this in mind we should really estimate $\mu + a_i + k\beta$, where k is some value in the range of x's in the experiment. With regard to treatment differences, $a_i - a_{i'}$, can be regarded as an estimate of $(\mu + a_i + k\beta) - (\mu + a_{i'} + k\beta)$, where k is in the range of the x's of the experiment.

6 Nonhomogenous Regressions

A still different covariate model is

$$y_{ij} = \mu + \beta_i x_{ij} + a_i + e_{ij}.$$

Note that in this model β is different from treatment to treatment. According to the rules for estimability $\mu + a_i$, $a_i - a_{i'}$, and β_i are all estimable. However, it is now obvious that $a_i - a_{i'}$ has no practical meaning as an estimate of treatment difference. We must specify what levels of x we assume to be present for each treatment. In terms of a treatment mean these are

$$\mu + a_i + k_i\beta_i$$

and

$$\mu + a_j + k_j\beta_j$$

and the difference is

$$a_i + k_i\beta_i - a_j - k_j\beta_j.$$

Suppose $k_i = k_j = k$. Then the treatment difference is

$$a_i - a_j + k(\beta_i - \beta_j),$$

and this is not invariant to the choice of k when $\beta_i \neq \beta_j$. In contrast when all $\beta_i = \beta$, the treatment difference is invariant to the choice of k.

Let us illustrate with two treatments.

Treatment	n_i	$y_{i.}$	$x_{i.}$	$\sum_j x_{ij}^2$	$\sum_j x_{ij}y_{ij}$
1	8	38	36	220	219
2	5	43	25	135	208

This gives least squares equations

$$\begin{pmatrix} 8 & 0 & 36 & 0 \\ & 5 & 0 & 25 \\ & & 220 & 0 \\ & & & 135 \end{pmatrix} \begin{pmatrix} \hat{\mu} + \hat{t}_1 \\ \hat{\mu} + \hat{t}_2 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{pmatrix} 38 \\ 43 \\ 219 \\ 208 \end{pmatrix}.$$

The solution is (1.0259, 12.1, .8276, -.7). Then the estimated difference, treatment 1 minus treatment 2 for various values for x, the same for each treatment, are as follows

x	Estimated Difference
0	-11.07
2	-8.02
4	-4.96
6	-1.91
8	1.15
10	4.20
12	7.26

It is obvious from this example that treatment differences are very sensitive to the average value of x.

7 The Usual One Way Random Model

Next we consider a model

$$\begin{aligned} y &= \mu + a_i + e_{ij}. \\ Var(\mathbf{a}) &= \mathbf{I}\sigma_a^2, \\ Var(\mathbf{e}) &= \mathbf{I}\sigma_e^2, \\ Cov(\mathbf{a}, \mathbf{e}') &= \mathbf{0}. \end{aligned}$$

In this case it is assumed that the levels of **a** in the sample are a random sample from an infinite population with var $\mathbf{I}\sigma_a^2$, and similarly for the sample of **e**. The experiment may have been conducted to do one of several things, estimate μ, predict **a**, or to estimate σ_a^2 and σ_e^2. We illustrate these with the following data.

Levels of a	n_i	$y_{i.}$
1	5	10
2	2	7
3	1	3
4	3	8
5	8	33

Let us estimate μ and predict $\mathbf{a}$ under the assumption that $\sigma_e^2/\sigma_a^2 = 10$. Then we need to solve these equations.

$$\begin{pmatrix} 19 & 5 & 2 & 1 & 3 & 8 \\ & 15 & 0 & 0 & 0 & 0 \\ & & 12 & 0 & 0 & 0 \\ & & & 11 & 0 & 0 \\ & & & & 13 & 0 \\ & & & & & 18 \end{pmatrix} \begin{pmatrix} \hat{\mu} \\ \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \\ \hat{a}_4 \\ \hat{a}_5 \end{pmatrix} = \begin{pmatrix} 61 \\ 10 \\ 7 \\ 3 \\ 8 \\ 33 \end{pmatrix}. \quad (12)$$

The solution is [3.137, -.379, .061, -.012, -.108, .439]. Note that $\sum \hat{a}_i = 0$. This could have been anticipated by noting that the sum of the last 4 equations minus the first equation gives

$$10 \sum \hat{a}_i = 0.$$

The inverse of the coefficient matrix is

$$\begin{pmatrix} .0790 & -.0263 & -.0132 & -.0072 & -.0182 & -.0351 \\ & .0754 & .0044 & .0024 & .0061 & .0117 \\ & & .0855 & .0012 & .0030 & .0059 \\ & & & .0916 & .0017 & .0032 \\ & & & & .0811 & .0081 \\ & & & & & .0712 \end{pmatrix}. \quad (13)$$

This matrix premultiplied by (0 1 1 1 1 1) equals (-1 1 1 1 1 1)(σ_a^2/σ_e^2). This is always a check on the inverse of the coefficient matrix in a model of this kind. From the inverse

$$\begin{aligned} Var(\hat{\mu}) &= .0790\, \sigma_e^2, \\ Var(\hat{a}_1 - a_1) &= .0754\, \sigma_e^2. \end{aligned}$$

$\hat{\mu}$ is BLUP of μ + the mean of all a in the infinite population. Similarly $\hat{a}_i$ is BLUP of a_i minus the mean of all a_i in the infinite population.

Let us estimate σ_a^2 by Method 1. For this we need $\sum_i y_{i.}^2/n_i$ and $y_{..}^2/n_.$ and their expectations. These are 210.9583 and 195.8421 with expectations, $19\sigma_a^2 + 5\sigma_e^2$ and $5.4211\sigma_a^2 + \sigma_e^2$ respectively ignoring 19 μ^2 in both.

$$\hat{\sigma}_e^2 = (\mathbf{y}'\mathbf{y} - 210.9583)/(19 - 5).$$

Suppose this is 2.8. Then $\hat{\sigma}_a^2 = .288$.

Let us next compute an approximate MIVQUE estimate using the prior $\sigma_e^2/\sigma_a^2 = 10$, the ratio used in the BLUP solution. We shall use $\hat{\sigma}_e^2 = 2.8$ from the least squares residual rather than a MIVQUE estimate. Then we need to compute $\hat{\mathbf{a}}'\hat{\mathbf{a}} = .35209$ and its expectation. The expectation is $trVar(\hat{\mathbf{a}})$. But $Var(\hat{\mathbf{a}}) = \mathbf{C}_a Var(\mathbf{r})\mathbf{C}_a'$, where $\mathbf{C}_a$ is the last 5 rows of the inverse of the mixed model equations (12), and $\mathbf{r}$ is the vector of right hand sides.

$$
\begin{aligned}
Var(\mathbf{r}) \;=\; & \begin{pmatrix} 5 & 2 & 1 & 3 & 8 \\ 5 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} 5 & 2 & 1 & 3 & 8 \\ 5 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix}' \sigma_a^2 + \\
& \begin{pmatrix} 19 & 5 & 2 & 1 & 3 & 8 \\ & 5 & 0 & 0 & 0 & 0 \\ & & 2 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 3 & 0 \\ & & & & & 8 \end{pmatrix} \sigma_e^2 \,.
\end{aligned}
$$

This gives

$$E(\hat{\mathbf{a}}'\hat{\mathbf{a}}) = .27163\,\sigma_a^2 + .06802\,\sigma_e^2,$$

and using $\hat{\sigma}_e^2 = 2.8$, we obtain $\hat{\sigma}_a^2 = .595$.

8 Finite Levels of a

Suppose now that the five a_i in the sample of our example of Section 7 comprise all of the elements of the population and that they are unrelated. Then

$$Var(\mathbf{a}) = \begin{pmatrix} 1 & & -.25 \\ & \ddots & \\ -.25 & & 1 \end{pmatrix} \sigma_a^2 \,.$$

Let us assume that $\sigma_e^2/\sigma_a^2 = 12.5$. Then the mixed model equations are the OLS equations premultiplied by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & .08 & -.02 & -.02 & -.02 & -.02 \\ & & .08 & -.02 & -.02 & -.02 \\ & & & .08 & -.02 & -.02 \\ & & & & .08 & -.02 \\ & & & & & .08 \end{pmatrix} . \tag{14}$$

This gives the same solution as that to (11). This is because σ_a^2 of the infinite model is $\frac{5}{4}$ times σ_a^2 of the finite model. See Section 15.9. Now $\hat{\mu}$ is a predictor of

$$\mu + \frac{1}{5}\sum_i a_i$$

and $\hat{a}_j$ is a predictor of

$$a_j - \frac{1}{5}\sum_i a_i.$$

Let us find the Method 1 estimate of σ_a^2 in the finite model. Again we compute $\sum_i y_{i.}^2/n_i$ and $y_{..}^2/n_{..}$. Then the coefficient of σ_e^2 in each of these is the same as in the infinite model, that is 5 and 1 respectively. For the coefficients of σ_a^2 we need the contribution of σ_a^2 to $Var(\text{rhs})$. This is

$$\begin{pmatrix} 5 & 2 & 1 & 3 & 8 \\ 5 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} 1 & & -\frac{1}{4} \\ & \ddots & \\ -\frac{1}{4} & & 1 \end{pmatrix} (\text{left matrix})'$$

$$= \begin{pmatrix} 38.5 & 7.5 & -4.5 & -3.5 & -3.0 & 42. \\ & 25.0 & -2.5 & -1.25 & -3.75 & -10. \\ & & 4.0 & -.5 & -1.5 & -4. \\ & & & 1. & -.75 & -2. \\ & & & & 9. & -6. \\ & & & & & 64. \end{pmatrix}. \qquad (15)$$

Then the coefficient of σ_a^2 in $\sum_i y_{i.}^2/n_i$ is $tr[dg(5,2,1,3,8)]^{-1}$ times the lower 5×5 submatrix of (15) = 19.0. The coefficient of σ_a^2 in $y_{..}^2/n_{..} = 38.5/19 = 2.0263$. Thus we need only the diagonals of (15). Assuming again that $\sigma_e^2 = 2.8$, we find $\hat{\sigma}_a^2 = .231$. Note that in the infinite model $\hat{\sigma}_a^2 = .288$ and that $\frac{5}{4}(.231) = .288$ except for rounding error. This demonstrates that we could estimate σ_a^2 as though we had an infinite model and estimate μ and predict $\mathbf{a}$ using $\hat{\sigma}_a^2/\hat{\sigma}_e^2$ in mixed model equations for the infinite model. Remember that the resulting inverse does not yield directly $Var(\hat{\mu})$ and $Var(\hat{\mathbf{a}} - \mathbf{a})$. For this pre- and post-multiply the inverse by

$$\frac{1}{5}\begin{pmatrix} 5 & 1 & 1 & 1 & 1 & 1 \\ 0 & 4 & -1 & -1 & -1 & -1 \\ 0 & -1 & 4 & -1 & -1 & -1 \\ 0 & -1 & -1 & 4 & -1 & -1 \\ 0 & -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & -1 & 4 \end{pmatrix}.$$

This is in accord with the idea that in the finite model $\hat{\mu}$ is BLUP of $\mu + \overline{a}.$ and $\hat{a}_i$ is BLUP of $a_i - \overline{a}.$.

9 One Way Random and Related Sires

We illustrate the use of the numerator relationship matrix in evaluating sires in a simple one way model,

$$\begin{aligned} y_{ij} &= \mu + s_i + e_{ij}. \\ Var(\mathbf{s}) &= \mathbf{A}\sigma_s^2, \\ Var(\mathbf{e}) &= \mathbf{I}\sigma_e^2, \\ Cov(\mathbf{s}, \mathbf{e}') &= \mathbf{0}, \\ \sigma_e^2/\sigma_s^2 &= 10. \end{aligned}$$

Then mixed model equations for estimation of μ and prediction of $\mathbf{s}$ are

$$\left(\begin{pmatrix} n_{.} & n_{1.} & n_{2.} & \ldots \\ n_{1.} & n_{1.} & 0 & \ldots \\ n_{2.} & 0 & & \\ \vdots & \vdots & & \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \ldots \\ 0 & & \mathbf{A}^{-1} & \sigma_e^2/\sigma_s^2 \\ 0 & & & \\ \vdots & & & \end{pmatrix} \right)$$

$$\begin{pmatrix} \hat{\mu} \\ \hat{s}_1 \\ \hat{s}_2 \\ \vdots \end{pmatrix} = (y_{..} \;\; y_{1.} \;\; y_{2.} \;\; \ldots)' \,. \tag{16}$$

We illustrate with the numerical example of section 7 but now with

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & .5 & .5 & 0 \\ & 1. & 0 & 0 & .5 \\ & & 1. & .25 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} .$$

The resulting mixed model equations are

$$\begin{pmatrix} 19 & 5 & 2 & 1 & 3 & 8 \\ & 65/3 & 0 & -20/3 & -20/3 & 0 \\ & & 46/3 & 0 & 0 & -20/3 \\ & & & 43/3 & 0 & 0 \\ & & & & 49/3 & 0 \\ & & & & & 64/3 \end{pmatrix} \begin{pmatrix} \hat{\mu} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{pmatrix} = \begin{pmatrix} 61 \\ 10 \\ 7 \\ 3 \\ 8 \\ 33 \end{pmatrix} . \tag{17}$$

The solution is (3.163, -.410, .232, -.202, -.259, .433). Note that $\sum_i \hat{s}_i \neq 0$ in contrast to the case in which $\mathbf{A} = \mathbf{I}$. Unbiased estimators of σ_e^2 and σ_s^2 can be obtained by computing Method 1 type quadratics, that is

$$\mathbf{y}'\mathbf{y} - \sum_i y_{i.}^2/n_i$$

and

$$\sum_i y_{i.}^2/n_i - \text{C.F.}$$

However, the expectations must take into account the fact that $Var(\mathbf{s}) \neq \mathbf{I}\sigma_s^2$, but rather $\mathbf{A}\sigma_s^2$. In a non-inbred population

$$E(\mathbf{y}'\mathbf{y}) = n_.(\sigma_s^2 + \sigma_e^2).$$

For an inbred population the expectation is

$$\sum_i n_i a_{ii} \sigma_a^2 + n_. \sigma_e^2,$$

where a_{ii} is the ith diagonal element of $\mathbf{A}$. The coefficients of σ_e^2 in $\sum y_{i.}^2/n_i$ and $y_{..}^2/n_.$ are the same as in an unrelated sample of sires. The coefficients of σ_s^2 require the diagonals of Var(rhs). For our example, these coefficients are

$$\begin{pmatrix} 5 & 2 & 1 & 3 & 8 \\ 5 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix} \mathbf{A} \text{ (left matrix)}'$$

$$= \begin{pmatrix} 140.5 & 35. & 12. & 4.25 & 17.25 & 72. \\ & 25. & 0 & 2.5 & 7.5 & 0 \\ & & 4. & 0 & 0 & 8. \\ & & & 1. & .75 & 0 \\ & & & & 9. & 0 \\ & & & & & 64. \end{pmatrix}. \quad (18)$$

Then the coefficient of σ_a^2 in $\sum_i y_{i.}^2/n_i$ is $tr(dg(0,\ 5^{-1},\ 2^{-1},\ 1,\ 3^{-1},\ 8^{-1}))$ times the matrix in (18) = 19. The coefficient of σ_a^2 in $y_{..}^2/n_. = 140.5/19 = 7.395$.

If we wanted an approximate MIVQUE we could compute rather than

$$\sum_i \frac{y_{1.}^2}{n_i} - \frac{y_{..}^2}{n_.}$$

of Method 1, the quadratic,

$$\hat{\mathbf{u}}'\mathbf{A}^{-1}\hat{\mathbf{u}} = .3602.$$

The expectation of this is

$$tr(\mathbf{A}^{-1}\ Var(\hat{\mathbf{s}})).$$
$$Var(\hat{\mathbf{s}}) = \mathbf{C}_s\ Var(\text{rhs})\ \mathbf{C}_s'.$$

$\mathbf{C}_s$ is the last 5 rows of the inverse of the mixed model coefficient matrix.

$$Var(\text{rhs}) = \text{Matrix (18) } \sigma_s^2 + \text{(OLS coefficient matrix) } \sigma_e^2.$$

Then

$$Var(\mathbf{s}) = \begin{pmatrix} .0788 & -.0527 & .0443 & .0526 & -.0836 \\ & .0425 & -.0303 & -.0420 & .0561 \\ & & .0285 & .0283 & -.0487 \\ & & & .0544 & -.0671 \\ & & & & .1014 \end{pmatrix} \sigma_s^2 +$$
$$\begin{pmatrix} .01284 & -.00774 & .00603 & .00535 & -.01006 \\ & .00982 & -.00516 & -.00677 & .00599 \\ & & .00731 & .00159 & -.00675 \\ & & & .01133 & -.00883 \\ & & & & .01462 \end{pmatrix} \sigma_e^2.$$

$\hat{\mathbf{s}}'\mathbf{A}^{-1}\hat{\mathbf{s}} = .36018$, with expectation $.05568\,\sigma_e^2 + .22977\,\sigma_s^2$. $\hat{\sigma}_e^2$ for approximate MIVQUE can be computed from

$$\mathbf{y}'\mathbf{y} - \sum_i y_{i.}^2/n_{i.}.$$