Chapter 14
Restricted Best Linear Prediction

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1 Restricted Selection Index

Kempthorne and Nordskog (1959) derived restricted selection index. The model and design assumed was that the record on the j^{th} trait for the i^{th} animal is

$$y_{ij} \;=\; \mathbf{x}_{ij}^{'}\boldsymbol{\beta} + u_{ij} + e_{ij}.$$

Suppose there are n animals and t traits. It is assumed that every animal has observations on all traits. Consequently there are nt records. Further assumptions follow. Let $\mathbf{u}_i$ and $\mathbf{e}_i$ be the vectors of dimension $t \times l$ pertaining to the i^{th} animal. Then it was assumed that

$$\begin{aligned} Var(\mathbf{u}_i) &= \mathbf{G}_0 \text{ for all } i = 1, \ldots, n, \\ Var(\mathbf{e}_i) &= \mathbf{R}_0 \text{ for all } i = 1, \ldots, n, \\ Cov(\mathbf{u}_i,\ \mathbf{u}_j^{'}) &= \mathbf{0} \text{ for all i}\neq\text{j}, \\ Cov(\mathbf{e}_i,\ \mathbf{e}_j^{'}) &= \mathbf{0} \text{ for all i}\neq\text{j}, \\ Cov(\mathbf{u}_i,\ \mathbf{e}_j^{'}) &= \mathbf{0} \text{ for all i, j}. \end{aligned}$$

Further $\mathbf{u}, \mathbf{e}$ are assumed to have a multivariate normal distribution and $\boldsymbol{\beta}$ is assumed known. This is the model for which truncation on selection index for $\mathbf{m}'\mathbf{u}_i$ maximizes the expectation of the mean of selected $\mathbf{m}'\mathbf{u}_i$, the selection index being the conditional mean and thus meeting the criteria for Cochran's (1951) result given in Section 5.1.

Kempthorne and Nordskog were interested in maximizing improvement in $\mathbf{m}'\mathbf{u}_i$ but at the same time not altering the expected value of $\mathbf{C}_0^{'}\mathbf{u}_i$ in the selected individuals, $\mathbf{C}_0^{'}$ being of dimension sxt and having s linearly independent rows. They proved that such a restricted selection index is

$$\mathbf{a}'\mathbf{y}^*,$$

where $\mathbf{y}^*$ = the deviations of $\mathbf{y}$ from their known means and $\mathbf{a}$ is the solution to

$$\begin{pmatrix} \mathbf{G}_0 + \mathbf{R}_0 & \mathbf{G}_0\mathbf{C} \\ \mathbf{C}'\mathbf{G}_0 & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \boldsymbol{\theta} \end{pmatrix} = \begin{pmatrix} \mathbf{G}_0\mathbf{m} \\ \mathbf{0} \end{pmatrix}. \quad (1)$$

This is a nice result but it depends upon knowing $\boldsymbol{\beta}$ and having unrelated animals and the same information on each candidate for selection. An extension of this to related animals, to unequal information, and to more general designs including progeny and sib tests is presented in the next section.

2 Restricted BLUP

We now return to the general mixed model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \mathbf{e},$$

where $\boldsymbol{\beta}$ is unknown, $Var(\mathbf{u}) = \mathbf{G}$, $Var(\mathbf{e}) = \mathbf{R}$ and $Cov(\mathbf{u}, \mathbf{e}') = \mathbf{0}$. We want to predict $\mathbf{k}'\boldsymbol{\beta} + \mathbf{m}'\mathbf{u}$ by $\mathbf{a}'\mathbf{y}$ where $\mathbf{a}$ is chosen so that $\mathbf{a}'\mathbf{y}$ is invariant to $\boldsymbol{\beta}$, $Var(\mathbf{a}'\mathbf{y} - \mathbf{k}'\boldsymbol{\beta} - \mathbf{m}'\mathbf{u})$ is minimum, and the expected value of $\mathbf{C}'\mathbf{u}$ given $\mathbf{a}'\mathbf{y} = \mathbf{0}$. This is accomplished by solving mixed model equations modified as in (2) and taking as the prediction $\mathbf{k}'\boldsymbol{\beta}^o + \mathbf{m}'\hat{\mathbf{u}}$.

$$\begin{pmatrix} \mathbf{X}'\mathbf{R}^{-1}\mathbf{X} & \mathbf{X}'\mathbf{R}^{-1}\mathbf{Z} & \mathbf{X}'\mathbf{R}^{-1}\mathbf{ZGC} \\ \mathbf{Z}'\mathbf{R}^{-1}\mathbf{X} & \mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z} + \mathbf{G}^{-1} & \mathbf{Z}'\mathbf{R}^{-1}\mathbf{ZGC} \\ \mathbf{C}'\mathbf{GZ}'\mathbf{R}^{-1}\mathbf{X} & \mathbf{C}'\mathbf{GZ}'\mathbf{R}^{-1}\mathbf{Z} & \mathbf{C}'\mathbf{GZ}'\mathbf{R}^{-1}\mathbf{ZGC} \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{\beta}^o \\ \hat{\mathbf{u}} \\ \boldsymbol{\theta} \end{pmatrix} = \begin{pmatrix} \mathbf{X}'\mathbf{R}^{-1}\mathbf{y} \\ \mathbf{Z}'\mathbf{R}^{-1}\mathbf{y} \\ \mathbf{C}'\mathbf{GZ}'\mathbf{R}^{-1}\mathbf{y} \end{pmatrix}. \quad (2)$$

It is easy to prove that $\mathbf{C}'\hat{\mathbf{u}} = \mathbf{0}$. Premultiply the second equation by $\mathbf{C}'\mathbf{G}$ and subtract from this the third equation. This gives $\mathbf{C}'\hat{\mathbf{u}} = \mathbf{0}$.

3 Application

Quaas and Henderson (1977) presented computing algorithms for restricted BLUP in an additively genetic model and with observations on a set of correlated animals. The algorithms permit missing data on some or all observations of animals to be evaluated. Two different algorithms are presented, namely records ordered traits within animals and records ordered animals within traits. They found that in this model absorption of $\boldsymbol{\theta}$ results in a set of equations with rank less than $r + q$, the rank of regular mixed model equations, where r = rank ($\mathbf{X}$) and q = number of elements in $\mathbf{u}$. The linear dependencies relate to the coefficients of $\boldsymbol{\beta}$ but not of $\mathbf{u}$. Consequently $\hat{\mathbf{u}}$ is unique, but care needs to be exercised in solving for $\boldsymbol{\beta}^o$ and in writing $\mathbf{K}'\boldsymbol{\beta} + \mathbf{m}'\mathbf{u}$, for $\mathbf{K}'\boldsymbol{\beta}$ must now be estimable under the augmented mixed model equations.