Chapter 11
MIVQUE of Variances and Covariances

C. R. Henderson

1984 - Guelph

The methods described in Chapter 10 for estimation of variances are quadratic, translation invariant, and unbiased. For the balanced design where there are equal numbers of observations in all subclasses and no covariates, equating the ANOVA mean squares to their expectations yields translation invariant, quadratic, unbiased estimators with minimum sampling variance regardless of the form of distribution, Albert (1976), see also Graybill and Wirtham (1956). Unfortunately, such an estimator cannot be derived in the unbalanced case unless $\mathbf{G}$ and $\mathbf{R}$ are known at least to proportionality. It is possible, however, to derive locally best, translation invariant, quadratic, unbiased estimators under the assumption of multivariate normality. This method is sometimes called MIVQUE and is due to C.R. Rao (1971). Additional pioneering work in this field was done by La Motte (1970,1971) and by Townsend and Searle (1971). By "locally best" is meant that if $\tilde{\mathbf{G}} = \mathbf{G}$ and $\tilde{\mathbf{R}} = \mathbf{R}$, the MIVQUE estimator has minimum sampling variance in the class of quadratic, unbiased, translation invariant estimators. $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{R}}$ are prior values of $\mathbf{G}$ and $\mathbf{R}$ that are used in computing the estimators. For the models which we have described in this book MIVQUE based on the mixed model equations is computationally advantageous. A result due to La Motte (1970) and a suggestion given to me by Harville have been used in deriving this type of MIVQUE algorithm. The equations to be solved are in (1).

$$\begin{pmatrix} \mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{X} & \mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{Z} \\ \mathbf{Z}'\tilde{\mathbf{R}}^{-1}\mathbf{X} & \mathbf{Z}'\tilde{\mathbf{R}}^{-1}\mathbf{Z} + \tilde{\mathbf{G}}^{-1} \end{pmatrix} \begin{pmatrix} \beta^o \\ \hat{\mathbf{u}} \end{pmatrix} = \begin{pmatrix} \mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{y} \\ \mathbf{Z}'\tilde{\mathbf{R}}^{-1}\mathbf{y} \end{pmatrix} \tag{1}$$

These are mixed model equations based on the model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \mathbf{e}. \tag{2}$$

We define $Var(\mathbf{u})$, $Var(\mathbf{e})$ and $Var(\mathbf{y})$ as in (2, 3, 4, 5, 6, 7, 8) of Chapter 10.

1 La Motte Result For MIVQUE

La Motte defined

$$Var(\mathbf{y}) = \mathbf{V} = \textstyle\sum_{i=1}^{k} \mathbf{V}_i \theta_i. \tag{3}$$

Then

$$\tilde{\mathbf{V}} = \sum\nolimits_{i=1}^{k} \mathbf{V}_i \tilde{\theta}_i, \tag{4}$$

where $\tilde{\theta}_i$ are prior values of θ_i. The θ_i are unknown parameters and the $\mathbf{V}_i$ are known matrices of order $n \times n$. He proved that MIVQUE of $\boldsymbol{\theta}$ is obtained by computing

$$(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}^o)'\tilde{\mathbf{V}}^{-1}\mathbf{V}_i\tilde{\mathbf{V}}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}^o),\ i = 1, \ldots,\ k, \tag{5}$$

equating these k quadratics to their expectations, and then solving for $\boldsymbol{\theta}$. $\boldsymbol{\beta}^o$ is any solution to equations

$$\mathbf{X}'\tilde{\mathbf{V}}^{-1}\mathbf{X}\boldsymbol{\beta}^o \ = \ \mathbf{X}'\tilde{\mathbf{V}}^{-1}\mathbf{y}. \tag{6}$$

These are GLS equations under the assumption that $\mathbf{V} \ = \ \tilde{\mathbf{V}}$.

2 Alternatives To La Motte Quadratics

In this section we show that other quadratics in $\mathbf{y} - \mathbf{X}\boldsymbol{\beta}^o$ exist which yield the same estimates as the La Motte formulation. This is important because there may be quadratics easier to compute than those of (5), and their expectations may be easier to compute.

Let the k quadratics of (5) be denoted by $\mathbf{q}$. Let $E(\mathbf{q}) = \mathbf{B}\boldsymbol{\theta}$, where $\mathbf{B}$ is $k \times k$. Then provided $\mathbf{B}$ is nonsingular, MIVQUE of $\boldsymbol{\theta}$ is

$$\hat{\boldsymbol{\theta}} = \mathbf{B}^{-1}\mathbf{q}. \tag{7}$$

Let $\mathbf{H}$ be any $k \times k$ nonsingular matrix. Compute a set of quadratics $\mathbf{Hq}$ and equate to their expectations.

$$E(\mathbf{Hq}) \ = \ \mathbf{H}E(\mathbf{q}) \ = \ \mathbf{HB}\theta. \tag{8}$$

Then an unbiased estimator is

$$\begin{aligned} \boldsymbol{\theta}^o &= (\mathbf{HB})^{-1}\mathbf{Hq} \\ &= \mathbf{B}^{-1}\mathbf{q} = \hat{\boldsymbol{\theta}}, \end{aligned} \tag{9}$$

the MIVQUE estimator of La Motte. Therefore, if we derive the La Motte quadratics, $\mathbf{q}$, for MIVQUE, we can find another set of quadratics which are also MIVQUE, and these are represented by $\mathbf{Hq}$, where $\mathbf{H}$ is nonsingular.

3 Quadratics Equal To La Motte's

The relationship between La Motte's model and ours is as follows

$$\mathbf{V} \text{ of LaMotte} = \mathbf{Z} \begin{pmatrix} \mathbf{G}_{11}g_{11} & \mathbf{G}_{12}g_{12} & \mathbf{G}_{13}g_{13} & \cdots \\ \mathbf{G}_{12}'g_{12} & \mathbf{G}_{22}g_{22} & \mathbf{G}_{23}g_{23} & \cdots \\ \mathbf{G}_{13}'g_{13} & \mathbf{G}_{23}'g_{23} & \mathbf{G}_{33}g_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \mathbf{Z}'$$

$$+\begin{pmatrix} \mathbf{R}_{11}r_{11} & \mathbf{R}_{12}r_{12} & \mathbf{R}_{13}r_{13} & \cdots \\ \mathbf{R}_{12}^{'}r_{12} & \mathbf{R}_{22}r_{22} & \mathbf{R}_{23}r_{23} & \cdots \\ \mathbf{R}_{13}^{'}r_{13} & \mathbf{R}_{13}^{'}r_{23} & \mathbf{R}_{33}r_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \tag{10}$$

$$= \mathbf{ZGZ}' + \mathbf{R}. \tag{11}$$

$$\text{or } \mathbf{V}_1\theta_1 = \mathbf{Z}\begin{pmatrix} \mathbf{G}_{11} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}\mathbf{Z}'g_{11},$$

$$\mathbf{V}_2\theta_2 = \mathbf{Z}\begin{pmatrix} \mathbf{0} & \mathbf{G}_{12} & \mathbf{0} & \cdots \\ \mathbf{G}_{12}^{'} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}\mathbf{Z}'g_{12}, \tag{12}$$

etc., and

$$\mathbf{V}_{b+1}\theta_{b+1} = \begin{pmatrix} \mathbf{R}_{11} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} r_{11},$$

$$\mathbf{V}_{b+2}\theta_{b+2} = \begin{pmatrix} \mathbf{0} & \mathbf{R}_{12} & \mathbf{0} & \cdots \\ \mathbf{R}_{12}^{'} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} r_{12}, \tag{13}$$

etc. Define the first $b(b+1)/2$ of (12) as $\mathbf{ZG}_{ij}^*\mathbf{Z}'$ and the last $c(c+1)/2$ of (13) as $\mathbf{R}_{ij}^*$. Then for one of the first $b(b+1)/2$ of La Motte's quadratic we have

$$(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}^o)'\tilde{\mathbf{V}}^{-1}\mathbf{ZG}_{ij}^*\mathbf{Z}'\tilde{\mathbf{V}}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}^o). \tag{14}$$

Write this as

$$(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}^o)'\tilde{\mathbf{V}}^{-1}\mathbf{Z}\tilde{\mathbf{G}}\tilde{\mathbf{G}}^{-1}\mathbf{G}_{ij}^*\tilde{\mathbf{G}}^{-1}\tilde{\mathbf{G}}\mathbf{Z}'\tilde{\mathbf{V}}^{-1}(\mathbf{y}\mathbf{X}\boldsymbol{\beta}^o). \tag{15}$$

This can be done because $\tilde{\mathbf{G}}\tilde{\mathbf{G}}^{-1} = \mathbf{I}$. Now note that $\tilde{\mathbf{G}}\mathbf{Z}'\tilde{\mathbf{V}}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}^o) = \hat{\mathbf{u}} =$ BLUP of $\mathbf{u}$ given $\mathbf{G} = \tilde{\mathbf{G}}$ and $\mathbf{R} = \tilde{\mathbf{R}}$. Consequently (15) can be written as

$$\hat{\mathbf{u}}'\tilde{\mathbf{G}}^{-1}\mathbf{G}_{ij}^*\tilde{\mathbf{G}}^{-1}\hat{\mathbf{u}}. \tag{16}$$

By the same type of argument the last c quadratics are

$$\hat{\mathbf{e}}'\tilde{\mathbf{R}}^{-1}\mathbf{R}_{ij}^*\tilde{\mathbf{R}}^{-1}\hat{\mathbf{e}}, \tag{17}$$

where $\hat{\mathbf{e}}$ is BLUP of $\mathbf{e}$ given that $\mathbf{G} = \tilde{\mathbf{G}}$ and $\mathbf{R} = \tilde{\mathbf{R}}$. Taking into account that

$$\mathbf{G} = \begin{pmatrix} \mathbf{G}_{11}g_{11} & \mathbf{G}_{12}g_{12} & \cdots \\ \mathbf{G}_{12}^{'}g_{12} & \mathbf{G}_{22}g_{22} & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

the matrices of the quadratics in $\hat{\mathbf{u}}$ can be computed easily. Let

$$\tilde{\mathbf{G}}^{-1} = \begin{pmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} & \mathbf{C}_{13} & \cdots \\ \mathbf{C}_{12}^{'} & \mathbf{C}_{22} & \mathbf{C}_{23} & \cdots \\ \mathbf{C}_{13}^{'} & \mathbf{C}_{23}^{'} & \mathbf{C}_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} = [\mathbf{C}_1 \ \mathbf{C}_2 \ \mathbf{C}_3 \ \ldots].$$

For example,

$$\mathbf{C}_2 = \begin{pmatrix} \mathbf{C}_{12} \\ \mathbf{C}_{22} \\ \mathbf{C}_{23}^{'} \\ \vdots \end{pmatrix}.$$

Then

$$\tilde{\mathbf{G}}^{-1}\mathbf{G}_{ii}^{*}\tilde{\mathbf{G}}^{-1} = \mathbf{C}_i\mathbf{G}_{ii}\mathbf{C}_i^{'}. \tag{18}$$

$$\tilde{\mathbf{G}}^{-1}\mathbf{G}_{ij}^{*}\tilde{\mathbf{G}}^{-1} = \mathbf{C}_i\mathbf{G}_{ij}\mathbf{C}_j^{'} + \mathbf{C}_j\mathbf{G}_{ij}^{'}\mathbf{C}_i \text{ for } i \neq j. \tag{19}$$

The quadratics in $\hat{\mathbf{e}}$ are like (18) and (19) with $\tilde{\mathbf{R}}^{-1}$, $\mathbf{R}_{ij}$ substituted for $\tilde{\mathbf{G}}^{-1}$, $\mathbf{G}_{ij}$ and with $\tilde{\mathbf{R}}^{-1} = [\mathbf{C}_1 \ \mathbf{C}_2 \ \ldots]$. For special cases these quadratics simplify considerably. First consider the case in which all $g_{ij} = 0$. Then

$$\mathbf{G} = \begin{pmatrix} \mathbf{G}_{11}g_{11} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{G}_{22}g_{22} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{G}_{33}g_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix},$$

and

$$\mathbf{G}^{-1} = \begin{pmatrix} \mathbf{G}_{11}^{-1}g_{11}^{-1} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{G}_{22}^{-1}g_{22}^{-1} & \cdots \\ \vdots & \vdots & \end{pmatrix}.$$

Then the quadratics in $\hat{\mathbf{u}}$ become

$$\hat{\mathbf{u}}_i^{'}\mathbf{G}_{ii}^{-1}g_{ii}^{-2}\hat{\mathbf{u}},$$

or an alternative is obviously

$$\hat{\mathbf{u}}_i^{'}\mathbf{G}_{ii}^{-1}\hat{\mathbf{u}}_i,$$

obtained by multiplying these quadratics by

$$\text{dg}(g_{11}^2, g_{22}^2, \ldots). \tag{20}$$

Similarly if all $r_{ij} = 0$, the quadratics in $\hat{\mathbf{e}}$ can be converted to

$$\hat{\mathbf{e}}_i^{'} \mathbf{R}_{ii}^{-1} \hat{\mathbf{e}}_i. \tag{21}$$

The traditional mixed model for variance components reduces to a particularly simple form. Because all $g_{ij} = 0$, for $i \neq j$, all $\mathbf{G}_{ii} = \mathbf{I}$, and $\mathbf{R} = \mathbf{I}$, the quadratics can be written as

$$\hat{\mathbf{u}}_i^{'} \hat{\mathbf{u}}_i \quad \text{i} = 1, \ldots, \text{ b}, \quad \text{and} \quad \hat{\mathbf{e}}' \hat{\mathbf{e}}.$$

Pre-multiplying these quadratics by

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \\ \sigma_e^2/\sigma_1^2 & \sigma_e^2/\sigma_2^2 & \cdots & 1 \end{pmatrix}$$

we obtain

$$\begin{pmatrix} \hat{\mathbf{u}}_1^{'} \hat{\mathbf{u}}_1 \\ \hat{\mathbf{u}}_2^{'} \hat{\mathbf{u}}_2 \\ \vdots \\ \hat{\mathbf{e}}' \hat{\mathbf{e}} \ + \ \sum_i \frac{\sigma_e^2}{\sigma_i^2} \ \hat{\mathbf{u}}_i^{'} \hat{\mathbf{u}}_i \end{pmatrix}.$$

But the last of these quadratics is $\mathbf{y}'\mathbf{y} - \mathbf{y}'\mathbf{X}\boldsymbol{\beta}^o - \mathbf{y}'\mathbf{Z}\hat{\mathbf{u}}$, or a quantity corresponding to the least squares residual. This is the algorithm described in Henderson (1973).

One might wish in this model to estimate σ_e^2 by the OLS residual mean square, that is,

$$\hat{\sigma}_e^2 \ = \ [\mathbf{y}'\mathbf{y} - (\boldsymbol{\beta}^o)'\mathbf{X}'\mathbf{y} - (\mathbf{u}^o)'\mathbf{Z}'\mathbf{y}]/[n - \text{rank}(\mathbf{X} \quad \mathbf{Z})],$$

where $\boldsymbol{\beta}^o$, $\mathbf{u}^o$ are some solution to OLS equations. If this is done, $\hat{\sigma}_e^2$ is not MIVQUE and neither are $\hat{\sigma}_i^2$, but they are probably good approximations to MIVQUE.

Another special case is the multiple trait model with additive genetic assumptions and elements of $\mathbf{u}$ for missing observations included. Ordering animals within traits,

$$\begin{aligned} \mathbf{y}' \ &= \ [\mathbf{y}_1^{'} \ \ \mathbf{y}_2^{'} \ldots]. \\ \mathbf{y}_i^{'} \ & \ \ \ \ \text{is the vector of records on the } i^{th} \text{ trait.} \\ \mathbf{u}' \ &= \ (\mathbf{u}_1^{'} \ \ \mathbf{u}_2^{'} \ldots). \\ \mathbf{u}_i \ & \ \ \ \ \text{is the vector of breeding values for the } i^{th} \text{ trait.} \\ \mathbf{e}' \ &= \ (\mathbf{e}_1^{'} \ \ \mathbf{e}_2^{'} \ldots). \end{aligned}$$

Every $\mathbf{u}_i$ vector has the same number of elements by including missing $\mathbf{u}$. Then

$$\mathbf{G} \;=\; \begin{pmatrix} \mathbf{A}g_{11} & \mathbf{A}g_{12} & \cdots \\ \mathbf{A}g_{12} & \mathbf{A}g_{22} & \cdots \\ \vdots & \vdots & \end{pmatrix} = \mathbf{A} * \mathbf{G}_0, \tag{22}$$

where

$$\mathbf{G}_0 = \begin{pmatrix} g_{11} & g_{12} & \cdots \\ g_{12} & g_{22} & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

is the additive genetic variance- covariance matrix for a non-inbred population, and $*$ denotes the direct product operation.

$$\mathbf{G}^{-1} = \begin{pmatrix} \mathbf{A}^{-1}g^{11} & \mathbf{A}^{-1}g^{12} & \cdots \\ \mathbf{A}^{-1}g^{12} & \mathbf{A}^{-1}g^{22} & \cdots \\ \vdots & \vdots & \end{pmatrix} \;=\; \mathbf{A}^{-1} * \mathbf{G}_0^{-1}. \tag{23}$$

Applying the methods of (16) to (18) and (19) the quadratics in $\hat{\mathbf{u}}$ illustrated for a 3 trait model are

$$\begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{12}' & \mathbf{B}_{22} \end{pmatrix} \begin{pmatrix} \hat{\mathbf{u}}_1'\mathbf{A}^{-1}\hat{\mathbf{u}}_1 \\ \hat{\mathbf{u}}_1'\mathbf{A}^{-1}\hat{\mathbf{u}}_2 \\ \hat{\mathbf{u}}_1'\mathbf{A}^{-1}\hat{\mathbf{u}}_3 \\ \hat{\mathbf{u}}_2'\mathbf{A}^{-1}\hat{\mathbf{u}}_2 \\ \hat{\mathbf{u}}_2'\mathbf{A}^{-1}\hat{\mathbf{u}}_3 \\ \hat{\mathbf{u}}_3'\mathbf{A}^{-1}\hat{\mathbf{u}}_3 \end{pmatrix}.$$

$$\mathbf{B}_{11} = \begin{pmatrix} g^{11}g^{11} & 2g^{11}g^{12} & 2g^{11}g^{13} \\ & 2g^{11}g^{22} + 2g^{12}g^{12} & 2g^{11}g^{23} + 2g^{12}g^{13} \\ & & 2g^{11}g^{33} + 2g^{13}g^{13} \end{pmatrix}.$$

$$\mathbf{B}_{12} = \begin{pmatrix} g^{12}g^{12} & 2g^{12}g^{13} & g^{13}g^{13} \\ 2g^{12}g^{22} & 2g^{12}g^{23} + 2g^{13}g^{22} & 2g^{13}g^{23} \\ 2g^{12}g^{23} & 2g^{12}g^{33} + 2g^{13}g^{23} & 2g^{13}g^{33} \end{pmatrix}.$$

$$\mathbf{B}_{22} = \begin{pmatrix} g^{22}g^{22} & 2g^{22}g^{23} & g^{23}g^{23} \\ & 2g^{22}g^{33} + 2g^{23}g^{23} & 2g^{23}g^{33} \\ & & g^{33}g^{33} \end{pmatrix}.$$

Premultiplying these quadratics in $\hat{\mathbf{u}}_i$ by the inverse of $\begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{12}' & \mathbf{B}_{22} \end{pmatrix}$ we obtain an equivalent set of quadratics that are

$$\hat{\mathbf{u}}_i'\mathbf{A}^{-1}\hat{\mathbf{u}}_j \text{ for i} = 1, \ldots, 3;\ \text{j} = \text{i}, \ldots, 3. \tag{24}$$

Similarly if there are no missing observations,

$$\mathbf{R} = \begin{pmatrix} \mathbf{I}\, r_{11} & \mathbf{I}\, r_{12} & \cdots \\ \mathbf{I}\, r_{12} & \mathbf{I}\, r_{22} & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

Then quadratics in $\hat{\mathbf{e}}$ are $\hat{\mathbf{e}}_i'\hat{\mathbf{e}}_j$ for $\mathrm{i} = 1, \ldots, \mathrm{t};\ \mathrm{j} = \mathrm{i}, \ldots, \mathrm{t}$.

4 Computation Of Missing $\hat{\mathbf{u}}$

In most problems, MIVQUE is easier to compute if missing $\mathbf{u}$ are included in $\hat{\mathbf{u}}$ rather than ignoring them. Section 3 illustrates this with $\mathbf{A}^{-1}$ being the matrix of all quadratics in $\hat{\mathbf{u}}$.

Three methods for prediction of elements of $\mathbf{u}$ not in the model for $\mathbf{y}$ were described in Chapter 5. Any of these can be used for MIVQUE. Probably the easiest is to include the missing ones in the mixed model equations.

5 Quadratics In $\hat{\mathbf{e}}$ With Missing Observations

When there are missing observations the quadratics in $\hat{\mathbf{e}}$ are easier to envision if we order the data by traits in animals rather than by animals in traits. Then $\mathbf{R}$ is block diagonal with the order of the i^{th} diagonal block being the number of traits recorded for the i^{th} animal. Now we do not need to store $\tilde{\mathbf{R}}^{-1}$ nor even all of the diagonal blocks. Rather we need to store only one block for each of the combinations of traits observed. For example, with 3 traits the possible combinations are

	Traits		
Combinations	1	2	3
1	X	X	X
2	X	X	-
3	X	-	X
4	-	X	X
5	X	-	-
6	-	X	-
7	-	-	X

There are $2^t - 1$ possible combinations for t traits. In the case of sequential culling the possible types are

	Traits		
Combinations	1	2	3
1	X	X	X
2	X	X	-
3	X	-	-

There are t possible combinations for t traits.

The block of $\tilde{\mathbf{R}}^{-1}$ for animals with the same traits measured will be identical. Thus if 50 animals have traits 1 and 2 recorded, there will be 50 identical 2×2 blocks in $\tilde{\mathbf{R}}^{-1}$, and only one of these needs to be stored.

The same principle applies to the matrices of quadratics in $\hat{\mathbf{e}}$. All of the quadratics are of the form $\hat{\mathbf{e}}_i^{'}\mathbf{Q}\hat{\mathbf{e}}_i$, where $\hat{\mathbf{e}}_i$ refers to the subvector of $\hat{\mathbf{e}}$ pertaining to the i^{th} animal. But animals with the same record combinations, will have identical matrices of quadratics for estimation of a particular variance or covariance. The computation of these matrices is simple. For a particular set of records let the block in $\tilde{\mathbf{R}}^{-1}$ be $\mathbf{P}$, which is symmetric and with order equal to the number of traits recorded. Label rows and columns by trait number. For example, suppose traits 1, 3, 7 are recorded. Then the rows (and columns) of $\mathbf{P}$ are understood to have labels 1, 3, 7. Let

$$\mathbf{P} \equiv (\mathbf{p}_1 \; \mathbf{p}_2 \; \mathbf{p}_3 \ldots),$$

where $\mathbf{p}_i$ is the i^{th} column vector of $\mathbf{P}$. Then the matrix of the quadratic for estimating r_{ii} is

$$\mathbf{p}_i\mathbf{p}_i^{'}. \tag{25}$$

The matrix for estimating r_{ij} $(i \neq j)$ is

$$\mathbf{p}_i\mathbf{p}_j^{'} + \mathbf{p}_i\mathbf{p}_j^{'}. \tag{26}$$

Let us illustrate with an animal having records on traits 2, 4, 7. The block of $\mathbf{R}$ corresponding to this type of information is

$$\begin{pmatrix} 6 & 4 & 3 \\ & 8 & 5 \\ & & 7 \end{pmatrix}.$$

Then the block corresponding to $\mathbf{R}^{-1}$ is the inverse of this, which is

$$\begin{pmatrix} .25410 & -.10656 & -.03279 \\ & .27049 & -.14754 \\ & & .26230 \end{pmatrix}.$$

Then the matrix for estimation of r_{22} is

$$\begin{pmatrix} .25410 \\ -.10656 \\ -.03279 \end{pmatrix} (.25410 \; - .10656 \; - .03279).$$

The matrix for computing r_{27} is

$$\begin{pmatrix} .25410 \\ -.10656 \\ -.03279 \end{pmatrix} (-.03279 \ \ -.14754 \ \ -.26230)$$

+ the transpose of this product.

6 Expectations Of Quadratics In $\hat{\mathbf{u}}$ And $\hat{\mathbf{e}}$

MIVQUE can be computed by equating certain quadratics in $\hat{\mathbf{u}}$ and in $\hat{\mathbf{e}}$ to their expectations. To find the expectations we need a g-inverse of the mixed model coefficient matrix with $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{R}}$, prior values, substituted for $\mathbf{G}$ and $\mathbf{R}$. The formulas for these expectations are in Section 6 of Chapter 10. It is obvious from these descriptions of expectations that extensive matrix products are required. However, some of the matrices have special forms such as diagonality, block diagonality, and symmetry. It is essential that these features be exploited. Also note that the trace of the products of several matrices can be expressed as the trace of the product of two matrices, say trace $(\mathbf{AB})$. Because only the sum of the diagonals of the product $\mathbf{AB}$ is required, it would be foolish to compute the off-diagonal elements. Some special computing algorithms are

$$\begin{aligned}
\text{trace } (\mathbf{AB}) &= \textstyle\sum_i \sum_j a_{ij} b_{ji} && (27)\\
&\quad \text{when } \mathbf{A} \text{ and } \mathbf{B} \text{ are nonsymmetric.} \\
\text{trace } (\mathbf{AB}) &= \textstyle\sum_i a_{ii} b_{ii} \ + \ 2\sum_i \sum_{j>i} a_{ij} b_{ij} && (28)\\
&\quad \text{when } \mathbf{A} \text{ and } \mathbf{B} \text{ are both symmetric.} \\
\text{tr } (\mathbf{AB}) &= \textstyle\sum_i a_{ii} b_{ii} && (29)\\
&\quad \text{when either } \mathbf{A} \text{ or } \mathbf{B} \text{ or both are diagonal.}
\end{aligned}$$

It is particularly important to take advantage of the form of matrices of quadratics in $\hat{\mathbf{e}}$ in animal models. When the data are ordered by traits within animals the necessary quadratics have the form $\sum_i \hat{\mathbf{e}}_i' \mathbf{Q} \hat{\mathbf{e}}_i$, where $\mathbf{Q}_i$ is a block of order equal to the number of traits observed in the i^{th} animal. Then the expectation of this quadratic is tr $\mathbf{Q}\ Var(\hat{\mathbf{e}}_i)$. Consequently we do not need to compute all elements of $Var(\hat{\mathbf{e}})$, but rather only the elements in blocks down the diagonal corresponding to the various $\mathbf{Q}$. In some cases, depending upon the form of $\mathbf{X}\boldsymbol{\beta}$, these blocks may be identical for animals with the same traits observed.

Many problems are such that the coefficient matrix is too large for computation of a g-inverse with present computers. Consequently we present in Section 7 an approximate MIVQUE based on computing an approximate g-inverse.

7 Approximate MIVQUE

MIVQUE for large data sets is prohibitively expensive with 1983 computers because a g-inverse of a very large coefficient matrix is required. Why not use an approximate g-inverse that is computationally feasible? This was the idea presented by Henderson(1980). The method is called "Diagonal MIVQUE" by some animal breeders. The feasibility of this method and the more general one presented in this section requires that an approximate g-inverse of $\mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{X}$ can be computed easily. First "absorb" $\boldsymbol{\beta}^o$ from the mixed model equations.

$$\begin{pmatrix} \mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{X} & \mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{Z} \\ \mathbf{Z}'\tilde{\mathbf{R}}^{-1}\mathbf{X} & \mathbf{Z}'\tilde{\mathbf{R}}^{-1}\mathbf{Z} + \tilde{\mathbf{G}}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}^o \\ \hat{\mathbf{u}} \end{pmatrix} = \begin{pmatrix} \mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{y} \\ \mathbf{Z}'\tilde{\mathbf{R}}^{-1}\mathbf{y} \end{pmatrix}. \tag{30}$$

This gives

$$[\mathbf{Z}'\mathbf{P}\mathbf{Z} + \tilde{\mathbf{G}}^{-1}]\ \hat{\mathbf{u}} = \mathbf{Z}'\mathbf{P}\mathbf{y} \tag{31}$$

where $\mathbf{P} = \tilde{\mathbf{R}}^{-1} - \tilde{\mathbf{R}}^{-1}\mathbf{X}(\mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{X})^{-}\mathbf{X}'\tilde{\mathbf{R}}^{-1}$, and $(\mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{X})^{-}$ is chosen to be symmetric. From the coefficient matrix of (31) one may see some simple approximate solution to $\hat{\mathbf{u}}$, say $\tilde{\mathbf{u}}$. Corresponding to this solution is a matrix $\tilde{\mathbf{C}}_{11}$ such that

$$\tilde{\mathbf{u}} = \tilde{\mathbf{C}}_{11}\mathbf{Z}'\mathbf{P}\mathbf{y} \tag{32}$$

Interpret $\tilde{\mathbf{C}}_{11}$ as an approximation to

$$\mathbf{C}_{11} \ = \ [\mathbf{Z}'\mathbf{P}\mathbf{Z} + \tilde{\mathbf{G}}^{-1}]^{-1}$$

Then given $\tilde{\mathbf{u}}$,

$$\tilde{\boldsymbol{\beta}} \ = \ (\mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{X})^{-}\ (\mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{y} - \mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{Z}\tilde{\mathbf{u}}).$$

Thus an approximate g-inverse to the coefficient matrix is

$$\tilde{\mathbf{C}} = \begin{pmatrix} \tilde{\mathbf{C}}_{00} & \tilde{\mathbf{C}}_{01} \\ \tilde{\mathbf{C}}_{10} & \tilde{\mathbf{C}}_{11} \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{C}}_{0} \\ \tilde{\mathbf{C}}_{1} \end{pmatrix}. \tag{33}$$

$$\begin{aligned}
\tilde{\mathbf{C}}_{00} &= (\mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{X})^{-} \ + \ (\mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{X})^{-}\mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{Z}\tilde{\mathbf{C}}_{11}\mathbf{Z}'\tilde{\mathbf{R}}^{-1}\mathbf{X}(\mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{X})^{-}. \\
\tilde{\mathbf{C}}_{01} &= (\mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{X})^{-}\mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{Z}\tilde{\mathbf{C}}_{11}. \\
\tilde{\mathbf{C}}_{10} &= \tilde{\mathbf{C}}_{11}\mathbf{Z}'\tilde{\mathbf{R}}^{-1}\mathbf{X}(\mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{X})^{-}.
\end{aligned}$$

This matrix post-multiplied by $\begin{pmatrix} \mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{y} \\ \mathbf{Z}'\tilde{\mathbf{R}}^{-1}\mathbf{y} \end{pmatrix}$ equals $\begin{pmatrix} \tilde{\boldsymbol{\beta}} \\ \tilde{\mathbf{u}} \end{pmatrix}$. Note that $\tilde{\mathbf{C}}_{11}$ may be non-symmetric.

What are some possibilities for finding an approximate easy solution to $\mathbf{u}$ and consequently for writing $\tilde{\mathbf{C}}_{11}$? The key to this decision is the pattern of elements of the matrix of (31). If the diagonal is large relative to off-diagonal elements of the same row for every row, setting $\tilde{\mathbf{C}}_{11}$ to the inverse of a diagonal matrix formed from the diagonals of the coefficient matrix is a logical choice. Harville suggested that for the two way mixed variance components model one might solve for the main effect elements of $\mathbf{u}$ by using only the diagonals, but the interaction terms would be solved by adjusting the right hand side for the previously estimated associated main effects and then dividing by the diagonal. This would result in a lower triangular $\tilde{\mathbf{C}}_{11}$.

The multi-trait equations would tend to exhibit block diagonal dominance if the elements of $\mathbf{u}$ are ordered traits within animals. Then $\tilde{\mathbf{C}}_{11}$ might well take the form

$$\begin{pmatrix} \mathbf{B}_1^{-1} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{B}_2^{-1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

where $\mathbf{B}_i^{-1}$ is the inverse of the i^{th} diagonal block, $\mathbf{B}_i$. Having solved for $\tilde{\mathbf{u}}$ and $\tilde{\boldsymbol{\beta}}$ and having derived $\tilde{\mathbf{C}}$ one would then proceed to compute quadratics in $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{e}}$ as in regular MIVQUE. Their expectations can be found as described in Section 7 of Chapter 10 except that $\tilde{\mathbf{C}}$ is substituted for $\mathbf{C}$.

8 MIVQUE (0)

MIVQUE simplifies greatly in the conventional variance components model if the priors are

$$\tilde{g}_{ii}/\tilde{r}_{11} \;=\; \hat{\sigma}_i^2/\hat{\sigma}_e^2 = 0 \text{ for all i} = 1, \ldots, \text{ b.}$$

Now

$$\tilde{\mathbf{V}} \;=\; \mathbf{I}\hat{\sigma}_e^2,\; \boldsymbol{\beta}^o = (\mathbf{X}'\mathbf{X})^-\mathbf{X}'\mathbf{y},$$

and

$$\begin{aligned} &(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}^o)'\ \tilde{\mathbf{V}}^{-1}\mathbf{Z}_i\mathbf{G}_{ii}\mathbf{Z}_i'\tilde{\mathbf{V}}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}^o) \\ &= \mathbf{y}'(\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}')'\mathbf{Z}_i\mathbf{Z}_i'(\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}')\mathbf{y}/\hat{\sigma}_e^4. \end{aligned} \tag{34}$$

Note that this, except for the constant, $\hat{\sigma}_e^4$, is simply the sum of squares of right hand sides of the OLS equations pertaining to $\mathbf{u}_i^o$ after absorbing $\boldsymbol{\beta}^o$. This is easy to compute, and the expectations are simple. Further, for estimation of σ_e^2 we derive the quadratic,

$$\mathbf{y}'\mathbf{y} - (\boldsymbol{\beta}^o)'\mathbf{X}'\mathbf{y}, \tag{35}$$

and the expectation of this is simple.

This method although simple to compute has been found to have large sampling variances when σ_i^2/σ_e^2 departs very much from 0, Quaas and Bolgiano(1979). Approximate MIVQUE involving diagonal $\tilde{\mathbf{C}}_{11}$ is not much more difficult and gives substantially smaller variances when $\sigma_i^2/\sigma_e^2 \;>\; 0$.

For the general model with $\tilde{\mathbf{G}} = \mathbf{0}$ the MIVQUE computations are effected as follows. This is an extension of MIVQUE(0) with $\mathbf{R} \neq \mathbf{I}\sigma_e^2$, and Var $(\mathbf{u}_i) \neq \mathbf{I}\sigma_i^2$. Absorb $\boldsymbol{\beta}^o$ from equations

$$\begin{pmatrix} \mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{X} & \mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{Z} \\ \mathbf{Z}'\tilde{\mathbf{R}}^{-1}\mathbf{X} & \mathbf{Z}'\tilde{\mathbf{R}}^{-1}\mathbf{Z} \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}^o \\ \mathbf{u}^o \end{pmatrix} = \begin{pmatrix} \mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{y} \\ \mathbf{Z}'\tilde{\mathbf{R}}^{-1}\mathbf{y} \end{pmatrix}. \tag{36}$$

Then compute

$$\mathbf{y}'\tilde{\mathbf{R}}^{-1}\mathbf{y} - (\mathbf{y}'\tilde{\mathbf{R}}^{-1}\mathbf{X})(\mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{X})^{-}\mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{y} \tag{37}$$

and $\mathbf{r}_i\mathbf{G}_{ij}\mathbf{r}_j$ i=1, ..., b; j=i, ..., b, where $\mathbf{r}_i$ = absorbed right hand side for $\mathbf{u}_i^o$ equations. Estimate r_{ij} from following quadratics

$$\hat{\mathbf{e}}_i\tilde{\mathbf{R}}_{ij}\hat{\mathbf{e}}_j$$

where

$$\hat{\mathbf{e}} \;=\; [\mathbf{I} - \mathbf{X}(\mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{X})^{-}\mathbf{X}'\tilde{\mathbf{R}}^{-1}]\mathbf{y}. \tag{38}$$

9 MIVQUE For Singular G

The formulation of (16) cannot be used if $\mathbf{G}$ is singular, neither can (18) if $\mathbf{G}_{ii}$ is singular, nor (25) if $\mathbf{A}$ is singular. A simple modification gets around this problem. Solve for $\hat{\boldsymbol{\alpha}}$ in (51) of Chapter 5. Then for (16) substitute

$$\hat{\boldsymbol{\alpha}}'\mathbf{G}_{ij}^*\hat{\boldsymbol{\alpha}}, \quad \text{where} \quad \hat{\mathbf{u}} = \tilde{\mathbf{G}}\hat{\boldsymbol{\alpha}} \tag{39}$$

For (20) substitute

$$\hat{\boldsymbol{\alpha}}_i'\mathbf{G}_{ii}\hat{\boldsymbol{\alpha}}_i. \tag{40}$$

For (25) substitute

$$\hat{\boldsymbol{\alpha}}_i'\mathbf{A}\hat{\boldsymbol{\alpha}}_j \tag{41}$$

See Section 16 for expectations of quadratics in $\hat{\boldsymbol{\alpha}}$.

10 MIVQUE For The Case $\mathbf{R} = \mathbf{I}\sigma_e^2$

When $\mathbf{R} \; = \mathbf{I}\sigma_e^2$ the mixed model equations can be written as

$$\begin{pmatrix} \mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{Z} \\ \mathbf{Z}'\mathbf{X} & \mathbf{Z}'\mathbf{Z} + \sigma_e^2\tilde{\mathbf{G}}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}^o \\ \hat{\mathbf{u}} \end{pmatrix} \;=\; \begin{pmatrix} \mathbf{X}'\mathbf{y} \\ \mathbf{Z}'\mathbf{y} \end{pmatrix}. \tag{42}$$

If $\boldsymbol{\beta}^o$ is absorbed, the equations in $\hat{\mathbf{u}}$ are

$$(\mathbf{Z}'\mathbf{P}\mathbf{Z} + \sigma_e^2 \tilde{\mathbf{G}}^{-1})\hat{\mathbf{u}} = \mathbf{Z}'\mathbf{P}\mathbf{y}, \tag{43}$$

where

$$\mathbf{P} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'.$$

Let

$$(\mathbf{Z}'\mathbf{P}\mathbf{Z} + \sigma_e^2 \tilde{\mathbf{G}}^{-1})^{-1} = \mathbf{C}. \tag{44}$$

Then

$$\begin{aligned}
\hat{\mathbf{u}} &= \mathbf{C}\mathbf{Z}'\mathbf{P}\mathbf{y}. && (45)\\
\hat{\mathbf{u}}'\mathbf{Q}_i\hat{\mathbf{u}} &= \mathbf{y}'\mathbf{P}\mathbf{Z}\mathbf{C}'\mathbf{Q}_i\mathbf{C}\mathbf{Z}'\mathbf{P}\mathbf{y} \\
&= \text{tr } \mathbf{C}'\mathbf{Q}_i\mathbf{C}\mathbf{Z}'\mathbf{P}\mathbf{y}\mathbf{y}'\mathbf{P}\mathbf{Z}. && (46)\\
E(\hat{\mathbf{u}}'\mathbf{Q}_i\hat{\mathbf{u}}) &= \text{tr } \mathbf{C}'\mathbf{Q}_i\mathbf{C}\ Var(\mathbf{Z}'\mathbf{P}\mathbf{y}). && (47)\\
Var(\mathbf{Z}'\mathbf{P}\mathbf{y}) &= \textstyle\sum_{i=1}^b \sum_{j=1}^b \mathbf{Z}'\mathbf{P}\mathbf{Z}_i\mathbf{G}_{ij}\mathbf{Z}_j'\mathbf{P}\mathbf{Z}g_{ij} \\
&\quad + \mathbf{Z}'\mathbf{P}\mathbf{P}\mathbf{Z}\sigma_e^2. && (48)
\end{aligned}$$

One might wish to obtain an approximate MIVQUE by estimating σ_e^2 from the OLS residual. When this is done, the expectation of the residual is $[n - r(\mathbf{X}\ \mathbf{Z})]\ \sigma_e^2$ regardless of the value of $\tilde{\mathbf{G}}$. This method is easier than true MIVQUE and has advantages in computation of sampling variances because the estimator of σ_e^2 is uncorrelated with the various $\hat{\mathbf{u}}'\mathbf{Q}\hat{\mathbf{u}}$. This method also is computable with absorption of $\boldsymbol{\beta}^o$.

A further simplification based on the ideas of Section 7, would be to look for some simple approximate solution to $\hat{\mathbf{u}}$ in (44). Call this solution $\tilde{\mathbf{u}}$ and the corresponding approximate g-inverse of the matrix of (44) $\tilde{\mathbf{C}}$. Then proceed as in (46) ... (48) except substitute $\tilde{\mathbf{u}}$ for $\hat{\mathbf{u}}$ and $\tilde{\mathbf{C}}$ for $\mathbf{C}$.

11 Sampling Variances

MIVQUE consists of computing $\hat{\mathbf{u}}'\mathbf{Q}_i\hat{\mathbf{u}}$, $i = 1, \ldots, b$, where b = number of elements of $\mathbf{g}$ to be estimated, and $\hat{\mathbf{e}}'\mathbf{Q}_j\hat{\mathbf{e}}$, $j = 1, \ldots, t$, where t = number of elements of $\mathbf{r}$ to be estimated. Let a g-inverse of the mixed model matrix be $\mathbf{C} \equiv \begin{pmatrix} \mathbf{C}_\beta \\ \mathbf{C}_u \end{pmatrix}$, and let $\mathbf{W} = (\mathbf{X} \quad \mathbf{Z})$.

Then

$$\begin{aligned}
\hat{\mathbf{u}} &= \mathbf{C}_u\mathbf{W}'\tilde{\mathbf{R}}^{-1}\mathbf{y}, \\
\hat{\mathbf{u}}'\mathbf{Q}\hat{\mathbf{u}} &= \mathbf{y}'\tilde{\mathbf{R}}^{-1}\mathbf{W}\mathbf{C}_u'\mathbf{Q}_i\mathbf{C}_u\mathbf{W}'\tilde{\mathbf{R}}^{-1}\mathbf{y} \equiv \mathbf{y}'\mathbf{B}_i\mathbf{y}, && (49)\\
\hat{\mathbf{e}} &= (\mathbf{I} - \mathbf{W}\mathbf{C}\mathbf{W}'\tilde{\mathbf{R}}^{-1})\mathbf{y},
\end{aligned}$$

and

$$\hat{\mathbf{e}}'\mathbf{Q}_j\hat{\mathbf{e}} = \mathbf{y}'[\mathbf{I} - \mathbf{WCW}'\tilde{\mathbf{R}}^{-1}]'\mathbf{Q}_j[\mathbf{I} - \mathbf{WCW}'\tilde{\mathbf{R}}^{-1}]\mathbf{y} \equiv \mathbf{y}'\mathbf{F}_j\mathbf{y}. \tag{50}$$

Let

$$E\begin{pmatrix} \mathbf{y}'\mathbf{B}_1\mathbf{y} \\ \vdots \\ \mathbf{y}'\mathbf{F}_1\mathbf{y} \\ \vdots \end{pmatrix} = \mathbf{P}\begin{pmatrix} \mathbf{g} \\ \mathbf{r} \end{pmatrix} = \mathbf{P}\boldsymbol{\theta}, \quad \text{where } \boldsymbol{\theta} = \begin{pmatrix} \mathbf{g} \\ \mathbf{r} \end{pmatrix}.$$

Then MIVQUE of $\boldsymbol{\theta}$ is

$$\mathbf{P}^{-1}\begin{pmatrix} \mathbf{y}'\mathbf{B}_1\mathbf{y} \\ \vdots \\ \mathbf{y}'\mathbf{F}_1\mathbf{y} \\ \vdots \end{pmatrix} = \begin{pmatrix} \mathbf{y}'\mathbf{H}_1\mathbf{y} \\ \mathbf{y}'\mathbf{H}_2\mathbf{y} \\ \vdots \end{pmatrix}. \tag{51}$$

Then

$$Var(\hat{\theta}_i) = 2\, tr[\mathbf{H}_i\, Var(\mathbf{y})]^2. \tag{52}$$

$$Cov(\hat{\theta}_i, \hat{\theta}_j) = 2\, tr\mathbf{H}_i\ [Var(\mathbf{y})]\ \mathbf{H}_j\ [Var(\mathbf{y})]. \tag{53}$$

These are of course quadratics in unknown elements of $\mathbf{g}$ and $\mathbf{r}$. A numerical solution is easier. Let $\tilde{\mathbf{V}} = Var(\mathbf{y})$ for some assumed values of $\mathbf{g}$ and $\mathbf{r}$. Then

$$Var(\hat{\theta}_i) = 2\, tr(\mathbf{H}_i\tilde{\mathbf{V}})^2. \tag{54}$$

$$Cov(\hat{\theta}_i, \hat{\theta}_j) = 2\, tr(\mathbf{H}_i\tilde{\mathbf{V}}\mathbf{H}_j\tilde{\mathbf{V}}). \tag{55}$$

If approximate MIVQUE is computed using $\tilde{\mathbf{C}}$ an approximation to $\mathbf{C}$, the computations are the same except that $\tilde{\mathbf{C}}$, $\tilde{\mathbf{u}}$, $\tilde{\mathbf{e}}$ are used in place of $\mathbf{C}$, $\hat{\mathbf{u}}$, $\hat{\mathbf{e}}$.

11.1 Result when σ_e^2 estimated from OLS residual

When $\mathbf{R} = \mathbf{I}\sigma_e^2$, one can estimate σ_e^2 by the residual mean square of OLS and an approximate MIVQUE obtained. The quadratics to be computed in addition to $\hat{\sigma}_e^2$ are only $\hat{\mathbf{u}}'\mathbf{Q}_i\hat{\mathbf{u}}$. Let

$$E\begin{pmatrix} \hat{\mathbf{u}}'\mathbf{Q}_i\hat{\mathbf{u}} \\ \vdots \\ \hat{\sigma}_e^2 \end{pmatrix} = \begin{pmatrix} \mathbf{P} & \mathbf{f} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{g} \\ \sigma_e^2 \end{pmatrix}.$$

Then

$$\begin{pmatrix} \hat{\mathbf{g}} \\ \hat{\sigma}_e^2 \end{pmatrix} = \begin{pmatrix} \mathbf{P} & \mathbf{f} \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} \hat{\mathbf{u}}'\mathbf{Q}_i\hat{\mathbf{u}} \\ \vdots \\ \hat{\sigma}_e^2 \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{u}}'\mathbf{H}_1\hat{\mathbf{u}} \\ \hat{\mathbf{u}}'\mathbf{H}_2\hat{\mathbf{u}} \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} s_1\hat{\sigma}_e^2 \\ s_2\hat{\sigma}_e^2 \\ \vdots \\ \hat{\sigma}_e^2 \end{pmatrix}. \tag{56}$$

Then

$$Var(\hat{g}_i) = 2\ tr[\mathbf{H}_i\ Var(\hat{\mathbf{u}})]^2 + s_i^2\ Var(\hat{\sigma}_e^2). \tag{57}$$

$$Cov(\hat{\mathbf{g}}_i, \hat{\mathbf{g}}_j) = 2\ tr[\mathbf{H}_i\ Var(\hat{\mathbf{u}})\ \mathbf{H}_j\ Var(\hat{\mathbf{u}})] + s_i s_j\ Var(\hat{\sigma}_e^2). \tag{58}$$

where

$$Var(\hat{\mathbf{u}}) = \mathbf{C}_u[Var(\mathbf{r})]\mathbf{C}_u',$$

and $\mathbf{r}$ equals the right hand sides of mixed model equations.

$$Var(\hat{\mathbf{u}}) = \mathbf{W}'\tilde{\mathbf{R}}^{-1}\mathbf{Z}\mathbf{G}\mathbf{Z}'\tilde{\mathbf{R}}^{-1}\mathbf{W} + \mathbf{W}'\tilde{\mathbf{R}}^{-1}\mathbf{R}\tilde{\mathbf{R}}^{-1}\mathbf{W}. \tag{59}$$

If $Var(\mathbf{r})$ is evaluated with the same values of $\mathbf{G}$ and $\mathbf{R}$ used in the mixed model equations, namely $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{R}}$, then

$$Var(\mathbf{r}) = \mathbf{W}'\tilde{\mathbf{R}}^{-1}\mathbf{Z}\tilde{\mathbf{G}}\mathbf{Z}'\tilde{\mathbf{R}}^{-1}\mathbf{W} + \mathbf{W}'\tilde{\mathbf{R}}^{-1}\mathbf{W}. \tag{60}$$

$$Var(\hat{\sigma}_e^2) = 2\sigma_e^4/[n - \text{rank}\ (\mathbf{W})], \tag{61}$$

where $\hat{\sigma}_e^2$ is the OLS residual mean square. This would presumably be evaluated for $\sigma_e^2 = \hat{\sigma}_e^2$.

12 Illustrations Of Approximate MIVQUE

12.1 MIVQUE with $\hat{\sigma}_e^2$ = OLS residual

We next illustrate several approximate MIVQUE using as $\hat{\sigma}_e^2$ the OLS residual. The same numerical example of treatments by sires in Chapter 10 is employed. In all of these we absorb $\boldsymbol{\beta}^o$ to obtain the equations already presented in (70) to (72) in Chapter 10. We use prior $\sigma_e^2/\sigma_s^2 = 10$, $\sigma_e^2/\sigma_{ts}^2 = 5$ as in Chapter 10. Then the equations in $\hat{\mathbf{s}}$ and $\hat{\mathbf{ts}}$ are those of (70) to (72) with 10 added to the first 4 diagonals and 5 to the last 10 diagonals. The inverse of this matrix is in (62), (63) and (64). This gives the solution

$$\hat{\mathbf{s}}' = [-.02966,\ .17793,\ .02693,\ -.17520].$$

$$\hat{\mathbf{ts}} = [.30280,\ .20042,\ .05299,\ -.55621,\ -.04723,\ .04635,\ .00088,\ -.31489,\ .10908,\ .20582].$$

$$(\hat{\mathbf{ts}})'\hat{\mathbf{ts}} = .60183,\ \hat{\mathbf{s}}'\hat{\mathbf{s}} = .06396.$$

Upper left 7×7

$$\begin{pmatrix} .0713 & .0137 & .0064 & .0086 & -.0248 & .0065 & .0070 \\ & .0750 & .0051 & .0062 & .0058 & -.0195 & .0052 \\ & & .0848 & .0037 & .0071 & .0048 & -.0191 \\ & & & .0815 & .0118 & .0081 & .0069 \\ & & & & .1331 & .0227 & .0169 \\ & & & & & .1486 & .0110 \\ & & & & & & .1582 \end{pmatrix} \quad (62)$$

Upper right 7×7 and (lower left 7×7)$'$

$$\begin{pmatrix} .0112 & -.0178 & .0121 & .0057 & -.0147 & .0088 & .0060 \\ .0085 & .0126 & -.0177 & .0050 & .0089 & -.0129 & .0040 \\ .0071 & .0061 & .0052 & -.0113 & -.0004 & .0001 & .0003 \\ -.0268 & -.0009 & .0004 & .0006 & .0063 & .0040 & -.0102 \\ .0273 & .0092 & -.0066 & -.0027 & .0081 & -.0045 & -.0036 \\ .0177 & -.0058 & .0071 & -.0014 & -.0039 & .0054 & -.0015 \\ .0138 & -.0025 & -.0011 & .0036 & -.0003 & .0004 & -.0001 \end{pmatrix} \quad (63)$$

Lower right 7×7

$$\begin{pmatrix} .1412 & -.0009 & .0005 & .0004 & -.0039 & -.0013 & .0052 \\ & .1489 & .0364 & .0147 & .0063 & -.0054 & -.0009 \\ & & .1521 & .0115 & -.0057 & .0055 & .0002 \\ & & & .1737 & -.0006 & -.0001 & .0007 \\ & & & & .1561 & .0274 & .0164 \\ & & & & & .1634 & .0092 \\ & & & & & & .1744 \end{pmatrix} \quad (64)$$

The expectation of $(\hat{\mathbf{ts}})'\hat{\mathbf{ts}}$ is

$$E[\mathbf{r}'\mathbf{C}_2'\mathbf{C}_2\mathbf{r}] = tr\mathbf{C}_2'\mathbf{C}_2 \; Var(\mathbf{r}),$$

where $\mathbf{r}$ = right hand sides of the absorbed equations, and $\mathbf{C}_2$ is the last 10 rows of the inverse above.

$$\begin{aligned} Var(\mathbf{r}) \;\; &= \;\; \text{matrix of (10.73) to (10.75) } \sigma^2_{ts} + \text{matrix of (10.76)} \\ & \quad \text{to (10.78) } \sigma^2_s + \text{matrix of (10.70) to (10.72) } \sigma^2_e. \end{aligned}$$

$\mathbf{C}_2'\mathbf{C}_2$ is in (65), (66), and (67). Similarly $\mathbf{C}_1'\mathbf{C}_1$ is in (68), (69), and (70) where $\mathbf{C}_2, \mathbf{C}_1$ refer to last 10 rows and last 4 rows of (62) to (64) respectively. This leads to expectations as follows

$$\begin{aligned} E(\hat{\mathbf{ts}}'\hat{\mathbf{ts}}) \;\; &= \;\; .23851 \; \sigma^2_e + .82246 \; \sigma^2_{ts} + .47406 \; \sigma^2_s. \\ E(\hat{\mathbf{s}}'\hat{\mathbf{s}}) \;\; &= \;\; .03587 \; \sigma^2_e + .11852 \; \sigma^2_{ts} + .27803 \; \sigma^2_s. \end{aligned}$$

Using $\hat\sigma_e^2 = .3945$ leads then to estimates,

$$\hat\sigma_{ts}^2 \;=\; .6815, \quad \hat\sigma_s^2 \;= -.1114.$$

Upper left 7×7

$$.01 \begin{pmatrix} .1657 & -.0769 & -.0301 & -.0588 & -.3165 & .0958 & .0982 \\ & .1267 & -.0167 & -.0331 & .0987 & -.2867 & .0815 \\ & & .0682 & -.0215 & .0977 & .0815 & -.2809 \\ & & & .1134 & .1201 & .1094 & .1012 \\ & & & & 1.9485 & .6920 & .5532 \\ & & & & & 2.3159 & .4018 \\ & & & & & & 2.5645 \end{pmatrix} \tag{65}$$

Upper right 7×7 and (lower left 7×7)'

$$.01 \begin{pmatrix} .1224 & -.2567 & .1594 & .0973 & -.2418 & .1380 & .1038 \\ .1065 & .1577 & -.2466 & .0889 & .1368 & -.2127 & .0758 \\ .1018 & .1007 & .0901 & -.1909 & -.0029 & .0012 & .0041 \\ -.3307 & -.0017 & -.0029 & .0046 & .1078 & .0759 & -.1837 \\ .8063 & .2207 & -.1480 & -.0726 & .2062 & -.1135 & -.0927 \\ .5902 & -.1364 & .1821 & -.0457 & -.1033 & .1505 & -.0473 \\ .4805 & -.0689 & -.0411 & .1101 & -.0059 & .0085 & -.0026 \end{pmatrix} \tag{66}$$

Lower right 7×7

$$.01 \begin{pmatrix} 2.1231 & -.0153 & .0071 & .0082 & -.0970 & -.0456 & .1426 \\ & 2.3897 & 1.0959 & .5144 & .1641 & -.1395 & -.0247 \\ & & 2.4738 & .4303 & -.1465 & .1450 & .0016 \\ & & & 3.0553 & -.0176 & -.0055 & .0231 \\ & & & & 2.5572 & .8795 & .5633 \\ & & & & & 2.7646 & .3559 \\ & & & & & & 3.0808 \end{pmatrix} \tag{67}$$

Upper right 7×7

$$.01 \begin{pmatrix} .5391 & .2087 & .1100 & .1422 & -.1542 & .0299 & .0510 \\ & .5879 & .0925 & .1109 & .0208 & -.1295 & .0429 \\ & & .7271 & .0705 & .0520 & .0382 & -.1522 \\ & & & .6764 & .0814 & .0613 & .0583 \\ & & & & .0840 & -.0145 & -.0199 \\ & & & & & .0510 & -.0091 \\ & & & & & & .0488 \end{pmatrix} \tag{68}$$

Upper right 7×7 and (lower left 7×7)'

$$
.01 \begin{pmatrix}
.0732 & -.1066 & .0656 & .0411 & -.0878 & .0484 & .0393 \\
.0658 & .0728 & -.1130 & .0402 & .0503 & -.0820 & .0317 \\
.0620 & .0464 & .0431 & -.0896 & -.0063 & .0017 & .0046 \\
-.2010 & -.0126 & .0043 & .0083 & .0438 & .0319 & -.0757 \\
-.0496 & .0548 & -.0360 & -.0187 & .0488 & -.0244 & -.0244 \\
-.0274 & -.0339 & .0450 & -.0111 & -.0220 & .0340 & -.0120 \\
-.0198 & -.0183 & -.0103 & .0286 & -.0006 & .0020 & -.0014
\end{pmatrix} \quad (69)
$$

Lower right 7×7

$$
.01 \begin{pmatrix}
.0968 & -.0025 & .0014 & .0012 & -.0261 & -.0116 & .0377 \\
 & .0514 & -.0406 & -.0108 & .0366 & -.0321 & -.0045 \\
 & & .0485 & -.0079 & -.0335 & .0335 & 0 \\
 & & & .0187 & -.0031 & -.0014 & .0045 \\
 & & & & .0335 & -.0219 & -.0117 \\
 & & & & & .0258 & -.0039 \\
 & & & & & & .0156
\end{pmatrix} \quad (70)
$$

12.2 Approximate MIVQUE using a diagonal g-inverse

An easy approximate MIVQUE involves solving for $\hat{\mathbf{u}}$ in the reduced equations by dividing the right hand sides by the corresponding diagonal coefficient. Thus the approximate $\mathbf{C}$, denoted by $\tilde{\mathbf{C}}$ is diagonal with diagonal elements the reciprocal of the diagonals of (10.70) to (10.72). This gives

$\tilde{\mathbf{C}}$ = dg (.0516, .0598, .0788, .0690, .1059, .1333, .1475, .1161, .1263, .1304, .1690, .1429, .1525, .1698)

and an approximate solution,

$\tilde{\mathbf{u}}'$ = (.0057, .2758, .0350, -.3563, .4000, .2889, .0656, -.7419, -.1263, .1304, 0, -.3810, .2203, .2076).

Then $(\hat{\mathbf{ts}})'\hat{\mathbf{ts}} = 1.06794$ with expectation,

$$.4024\ \sigma_e^2 \ + \ 1.5570\ (\sigma_{ts}^2 \ + \ \sigma_s^2).$$

Also $\hat{\mathbf{s}}'\hat{\mathbf{s}} = .20426$ with expectation,

$$.0871\ \sigma_e^2 \ + \ .3510\ \sigma_{ts}^2 \ + \ .7910\ \sigma_s^2.$$

Now

$\mathbf{C}_2\mathbf{C}_2' =$ dg (0, 0, 0, 0, 1.1211, 1.7778, 2.1768, 1.3486, 1.5959, 1.7013, 2.8566, 2.0408, 2.3269, 2.8836)/100,

and

$\mathbf{C}_1\mathbf{C}_1' =$ dg (.2668, .3576, .6205, .4756, 0, 0, 0, 0, 0, 0, 0, 0,, 0, 0, 0, 0, 0, 0)/100.

Consequently one would need to compute only the diagonals of (10.70) to (10.72), if one were to use this method of estimation.

12.3 Approximate MIVQUE using a block diagonal approximate g-inverse

Examination of (10.70) to (10.72) shows that a subset of coefficients, namely [s_j, ts_{1j}, $ts_{2j} \ldots$] tends to be dominant. Consequently one might wish to exploit this structure. If the $\hat{ts}_{ij}$ were reordered by i within j and the interactions associated with s_i placed adjacent to s_i, the matrix would exhibit block diagonal dominance. Consequently we solve for $\tilde{\mathbf{u}}$ in equations with the coefficient matrix zeroed except for coefficients of s_j and associated ts_{ij}, etc. blocks. This matrix is in (71, 72, and 73) below.

Upper left 7×7

$$\begin{pmatrix} 19.361 & 0 & 0 & 0 & 4.444 & 0 & 0 \\ & 16.722 & 0 & 0 & 0 & 2.5 & 0 \\ & & 12.694 & 0 & 0 & 0 & 1.778 \\ & & & 14.5 & 0 & 0 & 0 \\ & & & & 9.444 & 0 & 0 \\ & & & & & 7.5 & 0 \\ & & & & & & 6.778 \end{pmatrix} \tag{71}$$

Upper right 7×7 and (lower left 7×7)'

$$\begin{pmatrix} 0 & 2.917 & 0 & 0 & 2. & 0 & 0 \\ 0 & 0 & 2.667 & 0 & 0 & 1.556 & 0 \\ 0 & 2 & 0 & .917 & 0 & 0 & 0 \\ 3.611 & 0 & 0 & 0 & 0 & 0 & .889 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{72}$$

Lower right 7×7

$$\text{dg}\ (8.611,\ 7.9167,\ 7.6667,\ 5.9167,\ 7.0,\ 6.556,\ 5.889) \tag{73}$$

A matrix like (71) to (73) is easy to invert if we visualize the diagonal blocks with re-ordering. For example,

$$\begin{pmatrix} 19.361 & 4.444 & 2.917 & 2.000 \\ & 9.444 & 0 & 0 \\ & & 7.917 & 0 \\ & & & 7.000 \end{pmatrix}^{-1} = \begin{pmatrix} .0640 & -.0301 & -.0236 & -.0183 \\ & .1201 & .0111 & .0086 \\ & & .1350 & .0067 \\ & & & .1481 \end{pmatrix}.$$

This illustrates that only 4^2 or 3^2 order matrices need to be inverted. Also, each of those has a diagonal submatrix of order either 3 or 2. The resulting solution vector is

(-.0343, .2192, .0271, -.2079, .4162, .2158, .0585, -.6547, -.1137, .0542, -.0042, -.3711, .1683, .2389).

This gives $(\tilde{\mathbf{ts}})'\tilde{\mathbf{ts}} = .8909$ with expectation

$$.32515\ \sigma_e^2 \ + \ 1.22797\ \sigma_{ts}^2 \ + \ .79344\ \sigma_s^2,$$

and $\tilde{\mathbf{s}}'\tilde{\mathbf{s}} = .0932$ with expectation

$$.05120\ \sigma_e^2 \ + \ .18995\ \sigma_{ts}^2 \ + \ .45675\ \sigma_s^2.$$

12.4 Approximate MIVQUE using a triangular block diagonal approximate g-inverse

Another possibility for finding an approximate solution is to compute $\tilde{\mathbf{s}}$ by dividing the right hand side by the corresponding diagonal. Then $\tilde{\mathbf{ts}}$ are solved by adjusting the right hand side for the associated $\tilde{\mathbf{s}}$ and dividing by the diagonal coefficient. This leads to a block triangular coefficient matrix when $\mathbf{ts}$ are placed adjacent to $\mathbf{s}$. Without such re-ordering the matrix is as shown in (74), (75), and (76).

Upper left 7×7

$$\begin{pmatrix} 19.361 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 16.722 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12.694 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 14.5 & 0 & 0 & 0 \\ 4.444 & 0 & 0 & 0 & 9.444 & 0 & 0 \\ 0 & 2.5 & 0 & 0 & 0 & 7.5 & 0 \\ 0 & 0 & 1.778 & 0 & 0 & 0 & 6.778 \end{pmatrix} \quad (74)$$

Upper right 7×7 = null matrix

Lower left 7×7

$$\begin{pmatrix} 0 & 0 & 0 & 3.611 & 0 & 0 & 0 \\ 2.917 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2.667 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .917 & 0 & 0 & 0 & 0 \\ 2.0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.556 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .889 & 0 & 0 & 0 \end{pmatrix} \tag{75}$$

Lower right 7×7

$$\text{dg}\ (8.611,\ 7.917,\ 7.667,\ 5.917,\ 7.0,\ 6.556,\ 5.889) \tag{76}$$

This matrix is particularly easy to invert. The inverse has the zero elements in exactly the same position as the original matrix and one can obtain these by inverting triangular blocks illustrated by

$$\begin{pmatrix} 19.361 & 0 & 0 & 0 \\ 4.444 & 9.4444 & 0 & 0 \\ 2.917 & 0 & 7.917 & 0 \\ 2.000 & 0 & 0 & 7.000 \end{pmatrix}^{-1} = \begin{pmatrix} .0516 & 0 & 0 & 0 \\ -.0243 & .1059 & 0 & 0 \\ -.0190 & 0 & .1263 & 0 \\ -.0148 & 0 & 0 & .1429 \end{pmatrix}.$$

This results in the solution

(.0057, .2758, .0350, -.3563, .3973, .1970, .0564, -.5925, -.1284, .0345, -.0054, -.3826, .1549, .2613).

This gives $(\tilde{\mathbf{ts}})'\tilde{\mathbf{ts}} = .80728$ with expectation

$$.30426\ \sigma_e^2\ +\ 1.12858\sigma_{ts}^2\ +\ .60987\ \sigma_s^2,$$

and $\tilde{\mathbf{s}}'\tilde{\mathbf{s}} = .20426$ with expectation

$$.08714\ \sigma_e^2\ +\ .35104\sigma_{ts}^2\ +\ .79104\ \sigma_s^2.$$

13 An Algorithm for $\mathbf{R} = \mathbf{R}_*\sigma_e^2$ and Cov $(\mathbf{u}_i, \mathbf{u}_j') = \mathbf{0}$

Simplification of MIVQUE computations result if

$$\mathbf{R} = \mathbf{R}_*\sigma_e^2,\ \text{Var}\ (\mathbf{u}_i)\ =\ \mathbf{G}_{*i}\sigma_e^2\ ;\ \text{and Cov}\ (\mathbf{u}_i, \mathbf{u}_j')\ =\ \mathbf{0}.$$

$\mathbf{R}_*$ and the $\mathbf{G}_{*i}$ are known, and we wish to estimate σ_e^2 and the σ_i^2 . The mixed model equations can be written as

$$\begin{pmatrix} \mathbf{X}'\mathbf{R}_*^{-1}\mathbf{X} & \mathbf{X}'\mathbf{R}_*^{-1}\mathbf{Z}_1 & \mathbf{X}'\mathbf{R}_*^{-1}\mathbf{Z}_2 & \ldots \\ \mathbf{Z}_1'\mathbf{R}_*^{-1}\mathbf{X} & \mathbf{Z}_1'\mathbf{R}_*^{-1}\mathbf{Z}_1 + \mathbf{G}_{*1}^{-1}\alpha_1 & \mathbf{Z}_1'\mathbf{R}_*^{-1}\mathbf{Z}_2 & \ldots \\ \mathbf{Z}_2'\mathbf{R}_*^{-1}\mathbf{X} & \mathbf{Z}_2'\mathbf{R}_*^{-1}\mathbf{Z}_1 & \mathbf{Z}_2'\mathbf{R}_*^{-1}\mathbf{Z}_2 + \mathbf{G}_{*2}^{-1}\alpha_2 & \ldots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{\beta}^o \\ \hat{\mathbf{u}}_1 \\ \hat{\mathbf{u}}_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \mathbf{X}'\mathbf{R}_*^{-1}\mathbf{y} \\ \mathbf{Z}_1'\mathbf{R}_*^{-1}\mathbf{y} \\ \mathbf{Z}_2'\mathbf{R}_*^{-1}\mathbf{y} \\ \vdots \end{pmatrix}. \tag{77}$$

α_i = prior values of σ_e^2/σ_i^2. A set of quadratics equivalent to La Motte's are

$$\hat{\mathbf{e}}'\mathbf{R}_*^{-1}\hat{\mathbf{e}},\ \hat{\mathbf{u}}_i'\mathbf{G}_{*i}^{-1}\hat{\mathbf{u}}_i \quad (\text{i} = 1, 2, \ldots).$$

But because $\hat{\mathbf{e}}'\mathbf{R}_*^{-1}\hat{\mathbf{e}} = \mathbf{y}'\mathbf{R}_*^{-1}\mathbf{y}$ - (soln. vector)' (r.h.s. vector)

$$- \textstyle\sum_i \boldsymbol{\alpha}_i \hat{\mathbf{u}}_i'\mathbf{G}_{*i}^{-1}\hat{\mathbf{u}}_i,$$

an equivalent set of quadratics is

$$\mathbf{y}'\mathbf{R}_*^{-1}\mathbf{y}\ -\ (\text{soln. vector})'\ (\text{r.h.s. vector})$$

and

$$\hat{\mathbf{u}}_i'\mathbf{G}_{*i}^{-1}\hat{\mathbf{u}}_i \quad (\text{i} = 1, 2, \ldots).$$

14 Illustration Of MIVQUE In Multivariate Model

We illustrate several of the principles regarding MIVQUE with the following design

No. of Progeny			
	Sires		
Treatment	1	2	3
1	1	2	0
2	2	2	2

We assume treatments fixed with means t_1, t_2 respectively. The three sires are a random sample of unrelated sires from some population. Sire 1 had one progeny on treatment 1, and 2 different progeny on treatment 2, etc. for the other 2 sires. The sire and error variances are different for the 2 treatments. Further there is a non-zero error covariance between treatments. Thus we have to estimate g_{11} = sire variance for treatment l, g_{22} = sire variance for treatment 2, g_{12} = sire covariance, r_{11} = error variance for treatment 1, and r_{22} = error variance for treatment 2. We would expect no error covariance if the progeny are from unrelated dams as we shall assume. The record vector ordered by sires in treatments is [2, 3, 5, 7, 5, 9, 6, 8, 3].

We first use the basic La Motte method.

$$\mathbf{V_1} \text{ pertaining to } g_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 \\ & & & & & & 0 & 0 & 0 \\ & & & & & & & 0 & 0 \\ & & & & & & & & 0 \end{pmatrix}.$$

$$\mathbf{V_2} \text{ pertaining to } g_{12} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ & & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 \\ & & & & & & 0 & 0 & 0 \\ & & & & & & & 0 & 0 \\ & & & & & & & & 0 \end{pmatrix}.$$

$$\mathbf{V_3} \text{ pertaining to } g_{22} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 1 & 1 & 0 & 0 & 0 & 0 \\ & & & & 1 & 0 & 0 & 0 & 0 \\ & & & & & 1 & 1 & 0 & 0 \\ & & & & & & 1 & 0 & 0 \\ & & & & & & & 1 & 1 \\ & & & & & & & & 1 \end{pmatrix}.$$

$$\begin{aligned} \mathbf{V_4} \text{ pertaining to } r_{11} &= \text{dg } [1,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0]. \\ \mathbf{V_5} \text{ pertaining to } r_{22} &= \text{dg } [0,\ 0,\ 0,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1]. \end{aligned}$$

Use prior values of $g_{11} = 3$, $g_{12} = 2$, $g_{22} = 4$, $r_{11} = 30$, $r_{22} = 35$. Only the proportionality of these is of concern. Using these values

$$\tilde{\mathbf{V}} = \begin{pmatrix} 33 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ & 33 & 3 & 0 & 0 & 2 & 2 & 0 & 0 \\ & & 33 & 0 & 0 & 2 & 2 & 0 & 0 \\ & & & 39 & 4 & 0 & 0 & 0 & 0 \\ & & & & 39 & 0 & 0 & 0 & 0 \\ & & & & & 39 & 4 & 0 & 0 \\ & & & & & & 39 & 0 & 0 \\ & & & & & & & 39 & 4 \\ & & & & & & & & 39 \end{pmatrix}.$$

Computing $\tilde{\mathbf{V}}^{-1}\mathbf{V}_i\tilde{\mathbf{V}}^{-1}$ we obtain the following values for $\mathbf{Q}_i$ (i=1, ...,5). These are in the following table (times .001), only non-zero elements are shown.

Element	$\mathbf{Q}_1$	$\mathbf{Q}_2$	$\mathbf{Q}_3$	$\mathbf{Q}_4$	$\mathbf{Q}_5$
(1,1)	.92872	-.17278	.00804	.92872	.00402
(1,4),(1,5)	-.04320	.71675	-.06630	-.04320	-.03315
(2,2),(2,3)	.78781	-.14657	.00682	.94946	.00341
(2,6),(2,7)	-.07328	.66638	-.06135	-.03664	-.03068
(3,3)	.78781	-.14657	.00682	.94946	.00341
(3,6),(3,7)	-.07328	.66638	-.06135	-.03664	-.03068
(4,4)	.00201	-.06630	.54698	.00201	.68165
(4,5)	.00201	-.06630	.54698	.00201	-.13467
(5,5)	.00201	-.06630	.54698	.00201	.68165
(6,6)	.00682	-.12271	.55219	.00341	.68426
(6,7)	.00682	-.12271	.55219	.00341	-.13207
(7,7)	.00682	-.12271	.55219	.00341	.67858
(8,8),(9,9)	0	0	.54083	0	.67858
(8,9)	0	0	.54083	0	-.13775

We need $\mathbf{y} - \mathbf{X}\boldsymbol{\beta}^o$, $\boldsymbol{\beta}^o$ being a GLS solution. The GLS equations are

$$\begin{pmatrix} .08661 & -.008057 \\ -.008057 & .140289 \end{pmatrix} \boldsymbol{\beta}^o = \begin{pmatrix} .229319 \\ .862389 \end{pmatrix}.$$

The solution is [3.2368, 6.3333]. Then

$$
\begin{aligned}
\mathbf{y} - \mathbf{X}\boldsymbol{\beta}^o &= [-1.2368, -.2368,\ 1.7632,\ .6667,\ -1.3333,\ 2.6667, \\
&\quad -.3333,\ 1.6667,\ -3.3333]' \\
&= [\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{V}^{-1}]\mathbf{y} \equiv \mathbf{T}'\mathbf{y}.
\end{aligned}
$$

Next we need $\mathbf{T}'\mathbf{V}_i\mathbf{T}$ (i=1, ...,5) for the variance of $\mathbf{y} - \mathbf{X}\boldsymbol{\beta}^o$. These are

Element	$\mathbf{T}'\mathbf{V}_1\mathbf{T}$	$\mathbf{T}'\mathbf{V}_2\mathbf{T}$	$\mathbf{T}'\mathbf{V}_3\mathbf{T}$	$\mathbf{T}'\mathbf{V}_4\mathbf{T}$	$\mathbf{T}'\mathbf{V}_5\mathbf{T}$
(1,1)	.84017	-.03573	.00182	.63013	.00091
(1,2),(1,3)	-.45611	-.00817	.00182	-.34208	.00091
(1,4),(1,5)	0	.64814	.00172	0	.00086
(1,6),(1,7)	0	-.64814	.02928	0	.01464
(1,8),(1,9)					
(2,8),(2,9)	0	0	-.03101	0	-.01550
(3,8),(3,9)					
(2,2),(3,3)	.24761	.01940	.00182	.68571	.00091
(2,3)	.24761	.01940	.00182	-.31429	.00091
(2,4),(2,5)	0	-.35186	.00172	0	.00086
(3,4),(3,5)					
(2,6),(2,7)	0	.35186	.02928	0	.01464
(3,6),(3,7)					
(4,4),(5,5),(6,6)	0	0	.66667	0	.83333
(7,7),(8,8),(9,9)					
(4,5),(6,7),(8,9)	0	0	.66667	0	-.16667
(4,6),(4,7),(4,8)					
(4,9),(5,6),(5,7)	0	0	-.33333	0	-.16667
(5,8),(5,9),(6,8)					
(6,9),(7,8),(7,9)					

Taking all combinations of tr $\mathbf{Q}_i\mathbf{T}'\mathbf{V}_j\mathbf{T}$ for the expectation matrix and equating to $(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}^o)'\mathbf{Q}_i(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}^o)$ we have these equations to solve.

$$
\begin{pmatrix}
.00156056 & -.00029034 & .00001350 & .00117042 & .000006752 \\
 & .00372880 & -.00034435 & -.00021775 & -.00017218 \\
 & & .00435858 & .00001012 & .00217929 \\
 & & & .00198893 & .00000506 \\
 & & & & .00353862
\end{pmatrix}
\begin{pmatrix} g_{11} \\ g_{12} \\ g_{22} \\ r_{11} \\ r_{22} \end{pmatrix}
=
\begin{pmatrix} .00270080 \\ .00462513 \\ .00423360 \\ .00424783 \\ .01762701 \end{pmatrix}.
$$

This gives the solution [.500, 1.496, -2.083, 2.000, 6.333]. Note that the $\hat{g}_{ij}$ do not fall in the parameter space, but this is not surprising with such a small set of data.

Next we illustrate with quadratics in $\hat{u}_1, \ldots, \hat{u}_5$ and $\hat{e}_1, \ldots, \hat{e}_9$ using the same priors as before.

$$
\begin{aligned}
\mathbf{G}^*_{11} &= \text{dg } (1,\ 1,\ 0,\ 0,\ 0), \\
\mathbf{G}^*_{12} &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ & 0 & 0 & 1 & 0 \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}, \\
\mathbf{G}^*_{22} &= \text{dg } (0,\ 0,\ 1,\ 1,\ 1), \quad \tilde{\mathbf{R}} = \begin{pmatrix} 30\ \mathbf{I} & \mathbf{0} \\ \mathbf{0} & 35\ \mathbf{I} \end{pmatrix}. \\
\mathbf{R}^*_{11} &= \text{dg } (1,1,1,0,0,0,0,0,0), \\
\mathbf{R}^*_{22} &= \text{dg } (0,0,0,1,1,1,1,1,1), \\
\tilde{\mathbf{G}} &= \begin{pmatrix} 3 & 0 & 2 & 0 & 0 \\ & 3 & 0 & 2 & 0 \\ & & 4 & 0 & 0 \\ & & & 4 & 0 \\ & & & & 4 \end{pmatrix}.
\end{aligned}
$$

From these, the 3 matrices of quadratics in $\hat{\mathbf{u}}$ are

$$
\begin{pmatrix} .25 & 0 & -.125 & 0 & 0 \\ & .25 & 0 & -.125 & 0 \\ & & .0625 & 0 & 0 \\ & & & .0625 & 0 \\ & & & & 0 \end{pmatrix}, \quad
\begin{pmatrix} -.25 & 0 & .25 & 0 & 0 \\ & -.25 & 0 & .25 & 0 \\ & & -.1875 & 0 & 0 \\ & & & -.1875 & 0 \\ & & & & 0 \end{pmatrix},
$$

$$
\text{and} \quad \begin{pmatrix} .0625 & 0 & -.09375 & 0 & 0 \\ & .0625 & -.09375 & 0 & 0 \\ & & .140625 & 0 & 0 \\ & & & .140625 & 0 \\ & & & & .0625 \end{pmatrix}.
$$

Similarly matrices of quadratics in $\hat{\mathbf{e}}$ are

dg (.00111111, .00111111, .00111111, 0, 0, 0, 0, 0, 0),

and

dg (0, 0, 0, 1, 1, 1, 1, 1, 1)*.00081633.

The mixed model coefficient matrix is

$$\begin{pmatrix} .1 & 0 & .03333 & .06667 & 0 & 0 & 0 \\ & .17143 & 0 & 0 & .05714 & .05714 & .05714 \\ & & .53333 & 0 & -.25 & 0 & 0 \\ & & & .56667 & 0 & -.25 & 0 \\ & & & & .43214 & 0 & 0 \\ & & & & & .43214 & 0 \\ & & & & & & .30714 \end{pmatrix}.$$

The right hand side vector is

[.33333, 1.08571, .06667, .26667, .34286, .42857, .31429]′.

The solution is

[3.2368, 6.3333, -.1344, .2119, -.1218, .2769, -.1550].

Let the last 5 rows of the inverse of the matrix above = $\mathbf{C}_u$. Then

$$\begin{aligned} Var(\hat{\mathbf{u}}) &= \mathbf{C}_u\mathbf{W}'\tilde{\mathbf{R}}^{-1}\mathbf{Z}(\mathbf{G}^*_{11}g_{11} + \mathbf{G}^*_{12}g_{12} + \mathbf{G}^*_{22}g_{22})\mathbf{Z}'\tilde{\mathbf{R}}^{-1}\mathbf{W}\mathbf{C}_u' \\ &\quad +\mathbf{C}_u\mathbf{W}'\tilde{\mathbf{R}}^{-1}(\mathbf{R}^*_{11}r_{11} + \mathbf{R}^*_{22}r_{22})\tilde{\mathbf{R}}^{-1}\mathbf{W}\mathbf{C}_u' \end{aligned}$$

$$= \begin{pmatrix} .006179 & -.006179 & .003574 & -.003574 & 0 \\ & .006179 & -.003574 & .003574 & 0 \\ & & .002068 & -.002068 & 0 \\ & & & .002068 & 0 \\ & & & & 0 \end{pmatrix} g_{11}$$

$$+ \begin{pmatrix} .009191 & -.009191 & .012667 & -.012667 & 0 \\ & .009191 & -.012667 & .012667 & 0 \\ & & .011580 & -.011580 & 0 \\ & & & .011580 & 0 \\ & & & & 0 \end{pmatrix} g_{12}$$

$$+ \begin{pmatrix} .004860 & -.001976 & .010329 & -.004560 & -.005769 \\ & .004860 & -.004560 & .010329 & -.005769 \\ & & .021980 & -.010443 & -.011538 \\ & & & .021980 & -.011538 \\ & & & & .023076 \end{pmatrix} g_{22}$$

$$+ \begin{pmatrix} .004634 & -.004634 & .002681 & -.002681 & 0 \\ & .004634 & -.002681 & .002681 & 0 \\ & & .001551 & -.001551 & 0 \\ & & & .001551 & 0 \\ & & & & 0 \end{pmatrix} r_{11}$$

$$+\begin{pmatrix} .002430 & -.000988 & .005164 & -.002280 & -.002884 \\ & .002430 & -.002280 & .005164 & -.002884 \\ & & .010990 & -.005221 & -.005769 \\ & & & .010990 & -.005769 \\ & & & & .011538 \end{pmatrix} r_{22}.$$

$$\mathbf{e}' = [-1.1024,\ -4488,\ 1.5512,\ .7885,\ -1.2115,\ 2.3898,\ -.6102,\ 1.8217,\ -3.1783].$$

Let $\mathbf{C}$ be a g-inverse of the mixed model coefficient matrix, and $\mathbf{T} = \mathbf{I} - \mathbf{WCW}'\tilde{\mathbf{R}}^{-1}$. Then

$$Var(\hat{\mathbf{e}}) = \mathbf{T}(\mathbf{ZG}^*_{11}\mathbf{Z}'g_{11} + \mathbf{ZG}^*_{12}\mathbf{Z}'g_{12} + \mathbf{ZG}_{22}\mathbf{Z}'g_{22} + \mathbf{R}^*_{11}r_{11} + \mathbf{R}^*_{22}r_{22})\mathbf{T}'$$

$$= \begin{pmatrix} .702 & -.351 & -.351 & -.038 & -.038 & .031 & .038 & 0 & 0 \\ & .176 & .176 & .019 & .019 & -.019 & -.019 & 0 & 0 \\ & & .176 & .019 & .019 & -.019 & -.019 & 0 & 0 \\ & & & .002 & .002 & -.002 & -.002 & 0 & 0 \\ & & & & .002 & -.002 & -.002 & 0 & 0 \\ & & & & & .002 & .002 & 0 & 0 \\ & & & & & & .002 & 0 & 0 \\ & & & & & & & 0 & 0 \\ & & & & & & & & 0 \end{pmatrix} g_{11}$$

$$+ \begin{pmatrix} -.131 & .065 & .065 & .489 & .489 & -.489 & -.489 & 0 & 0 \\ & -.033 & -.033 & -.245 & -.245 & .245 & .245 & 0 & 0 \\ & & -.033 & -.245 & -.245 & .245 & .245 & 0 & 0 \\ & & & -.053 & -.053 & .053 & .053 & 0 & 0 \\ & & & & -.053 & .053 & .053 & 0 & 0 \\ & & & & & -.053 & -.053 & 0 & 0 \\ & & & & & & -.053 & 0 & 0 \\ & & & & & & & 0 & 0 \\ & & & & & & & & 0 \end{pmatrix} g_{12}$$

$$+ \begin{pmatrix} .006 & -.003 & -.003 & -.045 & -.045 & .045 & .045 & 0 & 0 \\ & .002 & .002 & .023 & .023 & -.023 & -.023 & 0 & 0 \\ & & .002 & .023 & .023 & -.023 & -.023 & 0 & 0 \\ & & & .447 & .447 & -.226 & -.226 & -.221 & -.221 \\ & & & & .447 & -.226 & -.226 & -.221 & -.221 \\ & & & & & .447 & .447 & -.221 & -.221 \\ & & & & & & .447 & -.221 & -.221 \\ & & & & & & & .442 & .442 \\ & & & & & & & & .442 \end{pmatrix} g_{22}$$

$$
+ \begin{pmatrix}
.527 & -.263 & -.263 & -.029 & -.029 & .029 & .029 & 0 & 0 \\
 & .632 & -.369 & .014 & .014 & -.014 & -.014 & 0 & 0 \\
 & & .632 & .014 & .014 & -.014 & -.014 & 0 & 0 \\
 & & & .002 & .002 & -.002 & -.002 & 0 & 0 \\
 & & & & .002 & -.002 & -.002 & 0 & 0 \\
 & & & & & .002 & .002 & 0 & 0 \\
 & & & & & & .002 & 0 & 0 \\
 & & & & & & & 0 & 0 \\
 & & & & & & & & 0
\end{pmatrix} r_{11}
$$

$$
+ \begin{pmatrix}
.003 & -.002 & -.002 & -.023 & -.023 & .023 & .023 & 0 & 0 \\
 & .001 & .001 & .011 & .011 & -.011 & -.011 & 0 & 0 \\
 & & .001 & .011 & .011 & -.011 & -.011 & 0 & 0 \\
 & & & .723 & -.277 & -.113 & -.113 & -.110 & -.110 \\
 & & & & .723 & -.113 & -.113 & -.110 & -.110 \\
 & & & & & .723 & -.277 & -.110 & -.110 \\
 & & & & & & .723 & -.110 & -.110 \\
 & & & & & & & .721 & -.279 \\
 & & & & & & & & .721
\end{pmatrix} r_{22}.
$$

Taking the traces of products of $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}_3$ with $Var(\hat{\mathbf{u}})$ and of $\mathbf{Q}_4$, $\mathbf{Q}_5$ with $Var(\hat{\mathbf{e}})$ we get the same expectations as in the La Motte method. Also the quadratics in $\hat{\mathbf{u}}$ and $\hat{\mathbf{e}}$ are the same as the La Motte quadratics in $(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}^o)$.

If $\hat{u}_6$ is included, the same quadratics and expectations are obtained. If $\hat{u}_6$ is included and we compute the following quadratics in $\hat{\mathbf{u}}$.

$$
\hat{\mathbf{u}}' \text{ dg } (1\ 1\ 1\ 0\ 0\ 0)\ \hat{\mathbf{u}}, \quad \hat{\mathbf{u}}' \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
 & 0 & 0 & 0 & 1 & 0 \\
 & & 0 & 0 & 0 & 1 \\
 & & & 0 & 0 & 0 \\
 & & & & 0 & 0 \\
 & & & & & 0
\end{pmatrix} \hat{\mathbf{u}},
$$

and $\hat{\mathbf{u}}'$ dg (0, 0, 0, 1, 1, 1) $\hat{\mathbf{u}}$ and equate to expectations we obtain exactly the same estimates as in the other three methods. We also could have computed the following quadratics in $\hat{\mathbf{e}}$ rather than the ones used, namely

$$
\hat{\mathbf{e}}' \text{ dg } (1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0)\hat{\mathbf{e}} \text{ and } \hat{\mathbf{e}}' \text{ dg } (0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1)\ \hat{\mathbf{e}}.
$$

Also we could have computed an approximate MIVQUE by estimating r_{11} from within sires in treatment 1 and r_{22} from within sires in treatment 2.

In most problems the "error" variances and covariances contribute markedly to computational labor. If no simplification of this computation can be effected, the La Motte quadratics might be used in place of quadratics in $\mathbf{e}$. Remember, however, that $\tilde{\mathbf{V}}^{-1}$ is usually a large matrix impossible to compute by conventional methods. But if $\tilde{\mathbf{R}}^{-1}$, $\tilde{\mathbf{G}}^{-1}$ and $(\mathbf{Z}'\tilde{\mathbf{R}}^{-1}\mathbf{Z} + \tilde{\mathbf{G}}^{-1})^{-1}$ are relatively easy to compute one can employ the results,

$$\tilde{\mathbf{V}}^{-1} = \tilde{\mathbf{R}}^{-1} - \tilde{\mathbf{R}}^{-1}\mathbf{Z}(\mathbf{Z}'\tilde{\mathbf{R}}^{-1}\mathbf{Z} + \tilde{\mathbf{G}}^{-1})^{-1}\mathbf{Z}\tilde{\mathbf{R}}^{-1}.$$

As already discussed, in most genetic problems simple quadratics in $\hat{\mathbf{u}}$ can be derived usually of the form

$$\hat{\mathbf{u}}_i^{'}\hat{\mathbf{u}}_j \text{ or } \hat{\mathbf{u}}_i\mathbf{A}^{-1}\hat{\mathbf{u}}_j.$$

Then these might be used with the La Motte ones for the r_{ij} rather than quadratics in $\hat{\mathbf{e}}$ for the r_{ij}. The La Motte quadratics are in $(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}^o)$, the variance of $\mathbf{y} - \mathbf{X}\boldsymbol{\beta}^o$ being

$$[\mathbf{I} - \mathbf{X}(\mathbf{X}'\tilde{\mathbf{V}}^{-1}\mathbf{X})^{-}\mathbf{X}'\tilde{\mathbf{V}}^{-1}]\mathbf{V}[\mathbf{I} - \mathbf{X}(\mathbf{X}'\tilde{\mathbf{V}}^{-1}\mathbf{X})^{-}\mathbf{X}'\tilde{\mathbf{V}}^{-1}]'.$$

Remember that $\tilde{\mathbf{V}} \neq \mathbf{V}$ in general, and $\mathbf{V}$ should be written in terms of g_{ij}, r_{ij} for purposes of taking expectations.

15 Other Types Of MIVQUE

The MIVQUE estimators of this chapter are translation invariant and unbiased. La Motte also presented other estimators including not translation invariant biased estimators and translation invariant biased estimators.

15.1 Not translation invariant and biased

The LaMotte estimator of this type is

$$\hat{\theta}_i = \tilde{\theta}_i \ (\mathbf{y} - \mathbf{X}\tilde{\boldsymbol{\beta}})'\tilde{\mathbf{V}}^{-1}(\mathbf{y} - \mathbf{X}\tilde{\boldsymbol{\beta}})/(\mathrm{n}+2),$$

where $\tilde{\theta}$, $\tilde{\boldsymbol{\beta}}$, and $\tilde{\mathbf{V}}$ are priors. This can also be computed as

$$\hat{\theta}_i = \tilde{\theta}_i \ [(\mathbf{y} - \mathbf{X}\tilde{\boldsymbol{\beta}})'\tilde{\mathbf{R}}^{-1}(\mathbf{y} - \mathbf{X}\tilde{\boldsymbol{\beta}}) - \hat{\mathbf{u}}'\mathbf{Z}'\tilde{\mathbf{R}}^{-1}(\mathbf{y} - \mathbf{X}\tilde{\boldsymbol{\beta}})]/(n+2),$$

where

$$\hat{\mathbf{u}} = (\mathbf{Z}'\tilde{\mathbf{R}}^{-1}\mathbf{Z} + \tilde{\mathbf{G}}^{-1})^{-1}\mathbf{Z}'\tilde{\mathbf{R}}^{-1}(\mathbf{y} - \mathbf{X}\tilde{\boldsymbol{\beta}}).$$

The lower bound on MSE of $\hat{\theta}_i$ is $2\tilde{\theta}_i^2/(\mathrm{n}+2)$, when $\tilde{\mathbf{V}}$, $\tilde{\boldsymbol{\beta}}$ are used as priors.

15.2 Translation invariant and biased

An estimator of this type is

$$\hat{\theta}_i \;=\; \tilde{\theta}_i \; (\mathbf{y} - \mathbf{X}\boldsymbol{\beta}^o)'\tilde{\mathbf{V}}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}^o)/(n - r + 2).$$

This can be written as

$$\tilde{\theta}_i \; [\mathbf{y}'\tilde{\mathbf{R}}^{-1}\mathbf{y} - (\boldsymbol{\beta}^o)'\mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{y} - \hat{\mathbf{u}}'\mathbf{Z}'\tilde{\mathbf{R}}^{-1}\mathbf{y}]/(\mathrm{n} - \mathrm{r} + 2).$$

$\boldsymbol{\beta}^o$ and $\hat{\mathbf{u}}$ are solution to mixed model equations with $\mathbf{G} = \tilde{\mathbf{G}}$ and $\mathbf{R} = \tilde{\mathbf{R}}$. The lower bound on MSE of $\hat{\theta}_i$ is $2\tilde{\theta}_i^2/(n-r+2)$ when $\tilde{\mathbf{V}}$ is used as the prior for $\mathbf{V}$. The lower bound on $\hat{\theta}_i$ for the translation invariant, unbiased MIVQUE is $2d_i$, when d_i is the i^{th} diagonal of $\mathbf{G}_0^{-1}$ and the ij^{th} element of $\mathbf{G}_0$ is $tr\mathbf{W}_0\mathbf{V}_i^*\mathbf{W}_0\mathbf{V}_i^*$ for

$$\mathbf{W}_0 = \tilde{\mathbf{V}}^{-1} - \tilde{\mathbf{V}}^{-1}\mathbf{X}(\mathbf{X}'\tilde{\mathbf{V}}^{-1}\mathbf{X})^{-}\mathbf{X}'\tilde{\mathbf{V}}^{-1}.$$

The estimators of sections 15.1 and 15.2 have the peculiar property that $\hat{\theta}_i/\hat{\theta}_j \;=\; \tilde{\theta}_i/\tilde{\theta}_j$. Thus the ratios of estimators are exactly proportional to the ratios of the priors used in the solution.

16 Expectations Of Quadratics In $\hat{\alpha}$

Let some g-inverse of (5.51) with priors on $\mathbf{G}$ and $\mathbf{R}$ be

$$\begin{pmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{12}' & \mathbf{C}_{22} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{pmatrix}$$

Then $\hat{\alpha} = \mathbf{C}_2\mathbf{r}$, where $\mathbf{r}$ is the right hand vector of (5.51), and

$$\begin{aligned} E(\hat{\boldsymbol{\alpha}}'\mathbf{Q}\hat{\boldsymbol{\alpha}}) &= tr\mathbf{Q}\; Var(\hat{\boldsymbol{\alpha}}) \\ &= tr\mathbf{Q}\mathbf{C}_2[Var(\mathbf{r})]\mathbf{C}_2'. \\ Var(\mathbf{r}) &= \begin{pmatrix} \mathbf{X}'\tilde{\mathbf{R}}^{-1}\mathbf{Z} \\ \tilde{\mathbf{G}}\mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z} \end{pmatrix} \mathbf{G}\; (\mathbf{Z}'\tilde{\mathbf{R}}^{-1}\mathbf{X} \quad \mathbf{Z}'\tilde{\mathbf{R}}^{-1}\mathbf{Z}\tilde{\mathbf{G}}) \\ &\quad + \begin{pmatrix} \mathbf{X}'\tilde{\mathbf{R}}^{-1} \\ \tilde{\mathbf{G}}\mathbf{Z}'\tilde{\mathbf{R}}^{-1} \end{pmatrix} \mathbf{R}\; (\tilde{\mathbf{R}}^{-1}\mathbf{X} \quad \tilde{\mathbf{R}}^{-1}\mathbf{Z}\tilde{\mathbf{G}}). \end{aligned} \tag{78}$$

When $\mathbf{R} = \mathbf{I}\sigma_e^2$ and $\mathbf{G} = \mathbf{G}_*\sigma_e^2$, $\hat{\boldsymbol{\alpha}}$ can be obtained from the solution to

$$\begin{pmatrix} \mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{Z}\mathbf{G}_* \\ \mathbf{G}_*\mathbf{Z}'\mathbf{X} & \mathbf{G}_*\mathbf{Z}'\mathbf{Z}\mathbf{G}_* + \mathbf{G}_* \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}^o \\ \hat{\boldsymbol{\alpha}} \end{pmatrix} = \begin{pmatrix} \mathbf{X}'\mathbf{y} \\ \mathbf{G}_*\mathbf{Z}'\mathbf{y} \end{pmatrix}. \tag{79}$$

In this case $\mathbf{C}_2$ is the last g rows of a g-inverse of (79).

$$\begin{aligned} Var(\mathbf{r}) &= \begin{pmatrix} \mathbf{X}'\mathbf{Z} \\ \tilde{\mathbf{G}}_*\mathbf{Z}'\mathbf{Z} \end{pmatrix} \mathbf{G} \ (\mathbf{Z}'\mathbf{X} \quad \mathbf{Z}'\mathbf{Z}\tilde{\mathbf{G}}_*)\sigma_e^2 \\ &+ \begin{pmatrix} \mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{Z}\tilde{\mathbf{G}}_* \\ \tilde{\mathbf{G}}_*\mathbf{Z}'\mathbf{X} & \tilde{\mathbf{G}}_*\mathbf{Z}'\mathbf{Z}\tilde{\mathbf{G}}_* \end{pmatrix} \sigma_e^2. \end{aligned} \tag{80}$$